Mathematical Reviews

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Mathematical Reviews

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LOGIC AND FOUNDATIONS

See also 5709.

5118:

Destouches, Jean-Louis. Descriptions opérationnelles en physique moderne. Synthese 10, 59–64.

This contrasts the axiomatic method with operational methods in scientific investigation. It is claimed that the axiomatic method suffices whenever, as in macrophysics, the role of the investigator can be eliminated. When the observer is an essential part of the experimental situation, as in microphysics, then operational methods will be needed. *O. Frink, Jr.* (University Park, Pa.)

5119:

Kamiński, Stanisław. Hobbes' theory of definition. Studia Logica 7 (1958), 43–69. (Polish. Russian and English summaries)

5120:

Kamiński, Stanisław. On the origin of mathematical induction. Studia Logica 7 (1958), 221–241. (Polish. Russian and English summaries)

5121:

Gál, L. Novak. A note on direct products. J. Symb. Logic 23 (1958), 1–6.

An algebra is an ordered set consisting of a class, a finite sequence of relations, and a finite sequence of operations. Given a finite sequence of decidable algebras, it is known that their direct product is decidable [S. Feferman, Formal consistency proof and interpretability of theories, Thesis, University of California, Berkeley, 1955]. The present paper presents two examples. Example A constructs a countable sequence of decidable algebras whose direct product is undecidable. In the sense of Example B, the former example illustrates also that the direct product of strongly decidable algebras need not be strongly decidable. Example B exhibits a countable sequence of strongly decidable algebras whose direct product is decidable, but not strongly decidable.
E. J. Cogan (Bronxville, N.Y.)

5122:

Hegenberg, Leonidas H. B. Table of the properties of the symbol ⊢ (turnstile). Soc. Parana. Mat. Anouári 4 (1957), 29–33. (Portuguese)

This is an expository article that reviews the properties of the theory of deduction which are in the form of derived rules in the calculus of propositions. After definitions of "immediate consequence", and "demonstration", a number of these rules are listed as properties of the turnstile. *E. J. Cogan* (Bronxville, N.Y.)

5123:

Robinson, Abraham. Outline of an introduction to mathematical logic. I. Canad. Math. Bull. 1 (1958), 41–54.

This is the first of a series of expository articles on the subject mentioned in the title. It is based on a course of lectures at the Edmonton seminar of the Canadian Mathematical Congress (1957). It treats the elements of the theory of truth functions, and a formulation of the deductive system of the Principia as revised by Bernays.
H. B. Curry (University Park, Pa.)

5124:

Robinson, Abraham. Outline of an introduction to mathematical logic. II. Canad. Math. Bull. 1 (1958), 113–127.

Continuation of Part I [reviewed above]. The author first shows that the system at the end of Part I is such that all theorems are tautologies and vice versa. He then formulates Boolean algebra and shows that classes of mutually equivalent truth functions form such an algebra. He ends by stating the definition of ideal in a Boolean algebra. *H. B. Curry* (University Park, Pa.)

5125:

Łoś, J.; and Suszko, R. Remarks on sentential logics. Nederl. Akad. Wetensch. Proc. Ser. A 61=Indag. Math. 20 (1958), 177–183.

This is a study of the methodology of systems based on propositional calculus (which the authors call "sentential logic"). It reminds one of Tarski's work on the methodology in 1930; but it is chiefly a condensed summary of articles published more recently by the authors in Polish. Let S be the free algebra, with propositional connectives as operations, generated by the propositional variables. A closure operation, Cn, on subsets of S, is called a "consequence" (Tarski's "Folgerung"); such a consequence Cn is called structural if $\varepsilon Cn(X)\subseteq Cn(\varepsilon X)$ for every endomorphism ε and subset X of S. Various properties and special cases of this notion are described, and assertions are stated concerning their interrelationships. Then an abstract form of the definition of logical matrix is given. Extending a theorem of Lindenbaum, the authors define for every matrix $\mathfrak{M}$ a consequence $\mathfrak{M}(X)$, and say that given a consequence $Cn(X)$, $\mathfrak{M}$ is adequate for Cn just when $Cn(X)=\mathfrak{M}(X)$ for every $X\subseteq S$. Their principal theorem is that for every structural consequence which is uniform, in a certain sense, there is an adequate matrix. Applications to classical intuitionistic, and positive logic are briefly discussed.
H. B. Curry (University Park, Pa.)

5126:

Lévy, A. The independence of various definitions of finiteness. Fund. Math. 46 (1958), 1–13.

The relative strength of seven different definitions of the notion of finiteness is established, where the basic set-theories considered are that of A. Mostowski [Fund. Math. 32 (1939), 201–252] and the same theory together with the total-ordering principle. Under the assumption of the axiom of choice, the seven definitions are equivalent. The counter-examples used in the independence proofs are found within certain models defined by Mostowski in the cited paper.
E. Mendelson (New York, N.Y.)

5127:

Burger, E. Eine Bemerkung zur Bernays-Gödel-Mengenlehre. Z. Math. Logik Grundlagen Math. **4** (1958), 178–179.

A short proof that Gödel's axiom B8 is superfluous.

L. N. Gál (New Haven, Conn.)

5128:

Hintikka, K. Jaakko J. Vicious circle principle and the paradoxes. J. Symb. Logic **22** (1957), 245–249.

In a previous paper [same J. **21** (1956), 225–245; MR **18**, 455] the author gave a new solution of the problem of the set theoretical antinomies. The solution consists of a modification of the principle of abstraction so as to insure that a set x will not be a member of itself, nor may it occur in the predicates by which it is defined.

In the present paper the author shows that, even in this modified form, the principle of abstraction gives rise to an antinomy. He concludes that the vicious circle principle of Russell and Whitehead is inadequate. Yet another reformulation of the principle of abstraction is discussed, but while it has not led to an antinomy so far, the set theory provided by it appears to be pathological.

A. Robinson (Jerusalem)

5129:

Guillaume, Marcel. Rapports entre calculs propositionnels modaux et topologie impliqués par certaines extensions de la méthode des tableaux sémantiques. Système S5 de Lewis. C. R. Acad. Sci. Paris **247** (1958), 1282–1283.

The methods previously developed by the author [C. R. Acad. Sci. Paris **246** (1958), 1140–1142, 2207–2210; MR **20** #2280; **21** #649] are now applied to Lewis' system S5 (which coincides with Von Wright's system M''), for which the following topological completeness theorem is established: A formula f is provable in S5 if and only if $\varphi(f)=E$ for every valuation φ of the set F of all S5-formulas over $\mathfrak{k}(E)$, for every finite set E and for the following choice of a closure operation in $\mathfrak{k}(E)$: $C\emptyset=\emptyset$, and if $\emptyset\neq X\subseteq\mathfrak{k}(E)$, then $CX=\mathfrak{k}(E)$.

E. W. Beth (Amsterdam)

5130:

Feferman, Solomon. Degrees of unsolvability associated with classes of formalized theories. J. Symb. Logic **22** (1957), 161–175.

Any formalized theory can be associated with the degree of unsolvability of the set of (Gödel numbers of) its valid sentences. Working with Tarski's formalized theories [A. Tarski, "Undecidable theories"; in collaboration with A. Mostowski, and R. M. Robinson; North-Holland Publ. Co., Amsterdam, 1953; MR **15**, 384] and Post's notion of degree of unsolvability [E. L. Post, Bull. Amer. Math. Soc. **50** (1944), 284–316; MR **6**, 29], the author has examined the usefulness of this association as a tool for studying various classes of theories.

For an arbitrary degree of unsolvability, there is an associated theory with no non-logical constants. (This strengthens the author's earlier result in Bull. Amer. Math. Soc. **62** (1956), 412 (abstract), where infinitely many non-logical constants were needed.) Consequently, for an arbitrary recursively enumerable degree of unsolvability, there is an associated axiomatizable theory. In the light of these results, the associated degree of unsolvability looks like a promising approach for studying the class of all theories or the class of all axiomatizable theories.

On the other hand, there is an important class of theories with creative sets of valid sentences. For metamathematical studies within this class, the above approach is useless, inasmuch as all theories are associated with the highest recursively enumerable degree. In showing the predominant role played by these "creative" theories, the author introduces the notion of "essential creativity": a consistent axiomatizable theory is essentially creative if every consistent axiomatizable extension with the same constants is creative. The crucial result is that, speaking now of axiomatizable theories and letting the notion of creativity take over the role of undecidability, Theorems 3 through 10 of Tarski, Mostowski, and Robinson, Part I, are true.

G. F. Rose (Santa Monica, Calif.)

5131:

Markov, A. A. Constructive functions. Trudy Mat. Inst. Steklov. **52** (1958), 315–348. (Russian)

The author gives a new, very general definition of the notion of a constructive function of real numbers (CFP). A recursive sequence $\varphi(n)$ of rational numbers is called regularly convergent if for all m and n such that $m\leqq n$, $|\varphi(m)-\varphi(n)|\leqq 2^{-m}$. $\varphi\geqslant a$ (a rational) means that there is a recursive function $g(n)$ such that $\varphi(n)\geqq a-2^{-b}$ for all $n\geqq g(b)$. $\varphi\sim\psi$ means $|\varphi-\psi|\leqslant 0$. A constructive real number (or a point) is an equivalence class of regularly convergent sequences with respect to $\sim$. If φ is a representing sequence of the point X, then $X=\varphi^*$; if $\varphi\leqslant a$, then $X\leqslant a$. In order to define constructive sequences of points (CSP), Gödel numbers are assigned to recursive sequences of rational numbers; the sequence with Gödel number e is denoted by $e^{\cdot}$. If $f(n)$ is a recursive function, such that $(f(n))^{\cdot}$ is regularly convergent for every n, then $\{X_n\}$, where $X_n=(f(n))^{\cdot *}$, is a CSP. $\{X_n\}$ converges to X if there is a recursive function $g(n)$, such that $|X_n-X|\leqslant 2^{-b}$ if $n\geqq g(b)$. Natural numbers d, e are called connected if $d^{\cdot}\sim e^{\cdot}$. A partial recursive function h which preserves connectedness, defines a CFP, h^0. If h^0 is defined for every member of the CSP $\{X_n\}$, $\{h^0X_n\}$ is a CSP. It is proved that a CFP cannot have a constructive point of discontinuity, in the following sense. The CSP $\{X_n\}$ is said to be apart from the point X, if $|X_n-X|\geqslant 2^{-a}$ for some a and for all n. X is a constructive point of discontinuity of the CFP, h^0, if for some CSP $\{X_n\}$, which converges to X, $\{h^0X_n\}$ is apart from h^0X. The proof utilizes the fact that the predicate $(\forall y)T_1(x, x, y)\neq 0$ is not recursive.

A. Heyting (Amsterdam)

5132:

Eršov, A. P. On operator algorithms. Dokl. Akad. Nauk SSSR **122** (1958), 967–970. (Russian)

A notion of "operator algorithm" is defined, and some proofs are sketched, showing that every partial recursive function is "realizable" by such an algorithm and that every normal algorithm of Markov has an equivalent operator algorithm. The treatment is very condensed, and, unfortunately, no particular examples of operator algorithms are given. The author states that operator algorithms will aid in the solution of many problems of "theoretical programming".

E. Mendelson (New York, N.Y.)

5133:

Dekker, J. C. E. Congruences in isols with a finite modulus. Math. Z. **70** (1958), 113–124.

A set is immune if it is infinite but has no infinite recursively enumerable subset. A set is isolated if it is finite or immune. A recursive equivalence type represented by an isolated set is called an isol. An arithmetic of isols is introduced with definitions of the operations $A+B$, $A\cdot B$, A^n (n finite), $A-B$, A/n, and the predicates

$A \leqq B$, $A \geqq B$, and $B|A$. Congruence modulo m is defined by $A \equiv B(\text{mod } m) \Leftrightarrow (\exists U)(\exists V)[A+mU=B+mV]$. The properties of congruences are established in section 3. In section 4, the notion of indecomposable isol is introduced and investigated. Section 5 is devoted to two proofs of an analogue of a version of Fermat's little theorem in isols. *E. J. Cogan* (Bronxville, N.Y.)

5134:

Dekker, J. C. E. The factorial function for isols. Math. Z. 70 (1958), 250–262.

This paper extends the functions $x!$ and

$$C(x, y)=\frac{x!}{y!(x-y)!}$$

to isols, and establishes some of the classical properties of these functions. *E. J. Cogan* (Bronxville, N.Y.)

5135:

da Costa, Newton Carneiro Affonso. Considerations on the Heyting calculus. Soc. Parana. Mat. Anuário 4 (1957), 42–46. (Portuguese)

This is an expository article which reviews elementary properties of the intuitionistic propositional calculus. After an intuitive discussion of the aims of Heyting's work, eleven tautologies are listed as the axioms of Heyting. The article is concluded with a discussion of the principle of excluded middle.

E. J. Cogan (Bronxville, N.Y.)

5136:

Février, P. Tendances constructives en logique moderne. Synthese 10, 52–58.

In section 1, the author contrasts the classical and formalist approach to mathematics to the constructive. The author introduces in section 2 three languages related to constructive mathematics: the language U, for general purposes of communication, the language C, which is limited and well defined, in which statements that constructions are completed may be formed, and L, the logic of constructions, which serves to establish the relations among completed constructions. Such a logic is introduced in section 3, which is followed by applications to the construction of a perpendicular from a point to a line, and of a sequence of rational approximations to $\sqrt{2}$. In the last section the criticism of the classical school is considered and various semi-constructive approaches to mathematics are suggested.

E. J. Cogan (Bronxville, N.Y.)

5137:

Rasiowa, H. $\mathcal{N}$-lattices and constructive logic with strong negation. Fund. Math. 46 (1958), 61–80.

The constructive propositional calculus with strong negation is a logical system which, by carrying two types of negation, mirrors the effects of intuitionistic and classical axioms. A $\mathcal{N}$-lattice is a distributive lattice satisfying certain axioms which give it the structure of the constructive propositional calculus with strong negation. In this paper methods similar to those of M. H. Stone [Časopis Pěst. Mat. Fys. 67 (1937), 1–25] are applied to $\mathcal{N}$-lattices to obtain a topological representation in terms of open sets of the bicompact T_0-space of all prime filters of the $\mathcal{N}$-lattice.

B. A. Galler (Ann Arbor, Mich.)

5138:

Porte, J. Une propriété du calcul propositionnel intuitionniste. Nederl. Akad. Wetensch. Proc. Ser. A 61=Indag. Math. 20 (1958), 362–365.

The paper is concerned with systems of propositional calculus based on the rules of modus ponens and substitution. Let C, I, J be, respectively, the classical, intuitionistic, and minimal systems. It is known that I has one and only one consistent and complete extension, whereas J has at least two. The author shows that if S is an extension of J which has C as its unique consistent complete extension, then S is an extension of I.

H. B. Curry (University Park, Pa.)

5139:

***Pickert, Günter. Ebene Inzidenzgeometrie: Beispiele zur Axiomatik mit einer Einführung in die formale Logik.** Schriftenreihe zur Mathematik, Heft 8. Otto Salle Verlag, Frankfurt am Main–Hamburg, 1958. 92 pp. DM 6.40.

Verf. hat sich in diesem Buch zum Ziel gesetzt zu zeigen, was ein mathematischer Beweis sei. Zu diesem Zweck nimmt Verf. die Axiome der affinen ebenen Inzidenzgeometrie als Ausgangspunkt und dringt vor bis zu einigen ersten endlichen Modellen dieser Geometrie, die er an Hand von Systemen orthogonaler lateinischer Quadrate erläutert. Er modifiziert die obengenannten Axiome zu den Axiomen der projektiven ebenen Inzidenzgeometrie und leitet einige Sätze dieser Geometrie ab. Er führt auch "interne" (d.h. nicht den Begriff der reellen Zahl benutzende) Koordinaten in affinen Ebenen ein. Im Zusammenhang damit entwickelt Verf. noch die analytische Geometrie aus dem Satz von Desargues im Rahmen der ebenen Inzidenzgeometrie und stellt damit den Zusammenhang mit dem algebraischen Begriff des Schiefkörpers her. Schliesslich zeigt Verf. im letzten Abschnitt, welche logische Regeln in seinen mathematischen Beweisen verwandt wurden.

Das Buch ist bestimmt für Schüler der Oberstufe, insbesondere für diejenigen Schüler, die im späteren Beruf nichts mehr mit Mathematik zu tun haben werden, für Studenten von nicht-mathematischen Fächern (z.B. der Philosophie), die aus irgendeinen Grunde Interesse für Mathematik haben, und für Lehrer der Mathematik. Es scheint aber dem Ref., dass es besonders geeignet ist, Mathematiker, die nicht Geometer sind, in die Geometrie einzuführen. Denn für die zuerst genannten Kategorien von Lesern ist das Buch wohl zu schwierig. Man muss schon sehr scharf aufpassen, wenn man den Beweisen des Verfassers folgen soll. Referent hat bisher kein einziges mathematisches Buch gesehen, das das richtige Gleichgewicht zwischen Strenge und Vollständigkeit der Darstellung einerseits und den Bedürfnissen und Grenzen der nicht in höherer Mathematik bewanderter Leser andererseits, hält. Auch dieses Buch erfüllt diesen Zweck nicht. Es mag aber sein, dass der Leser sich in einer Weise, die seinen Bedürfnissen entspricht, auf Grund dieses Buches zurechtlegen kann, was ein mathematischer Beweis sei, auch ohne das Buch restlos verstanden zu haben. Dabei ist sich Ref. bewusst, dass es ungleich leichter ist, zu kritisieren, als besser zu machen.

Aus der Darstellung des Verf. kann der Leser entnehmen, wie kompliziert die "endliche" Mathematik sein kann. Er kommt so zur Einsicht, dass die Einführung des Unendlichen in die Mathematik mit den Zweck hat, dieselbe zu vereinfachen. Das Buch vermittelt auch die Einsicht, dass das Rechnen mit Koordinaten auch ohne jeden Massbegriff möglich ist. *B. Germansky* (Berlin)

5140:

Kemeny, John G. Undecidable problems of elementary number theory. Math. Ann. 135 (1958), 160–169.

After an extensive and informative discussion of the construction of non-standard models of arithmetic and of

the bearing of such constructions upon questions of independence, the author constructs a non-standard model M_1 which contains 'abnormal' numbers, that is, individuals divisible by every positive integer. Then a partial system M_2 is considered which can be characterised as the closure under addition and multiplication of the subset of M_1 containing all positive integers and all abnormal numbers. For M_2 both the prime n-tuple conjectures (for $n \geqq 3$) and Goldberg's conjecture are false. Thus if M_2 is a model of arithmetic, these conjectures cannot be provable in arithmetic. However, the question as to whether M_2 is or is not a model of arithmetic is left open. *E. W. Beth* (Amsterdam)

5141:

Gandy, R. O. Note on a paper of Kemeny's. Math. Ann. 136 (1958), 466.

The author shows that, whenever n is a prime number and β is an unnatural number, none of the numbers $n^\beta + k$ (k a rational integer) can be an abnormal number, thus answering a question raised by J. G. Kemeny [#5140 above]. *E. W. Beth* (Amsterdam)

5142:

Trahtenbrot, B. A. The synthesis of logical nets whose operators are described in terms of one-place predicate calculus. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 646–649. (Russian)

The relation between the binary output Z and the binary input $X_1, \cdots, X_n$ of a computer with memory elements $\Gamma_1, \cdots, \Gamma_h$ can be described by a scheme of the form (i):

$$Z(t) = \Phi[X_1(t), \cdots, X_n(t), \Gamma_1(t), \cdots, \Gamma_h(t)],$$
$$\Gamma_\nu(t+1) = \Psi_\nu[X_1(t), \cdots, X_n(t), \Gamma_1(t), \cdots, \Gamma_h(t)],$$
$$\Gamma_\nu(1) = \sigma_\nu,$$

where Φ and Ψ_ν are functions in the propositional calculus, or equivalently by the formula

$$(E\Gamma_1) \cdots (E\Gamma_h)\{\Phi[X_1(t), \cdots, \Gamma_h(t)] \bigwedge_{\nu=1}^{h} (\tau)_{\tau \leq t}[(\tau = 1 \wedge \Gamma_\nu^{\sigma_\nu}(\tau)$$
$$\vee \ (E\sigma)_{\sigma<\tau}(\sigma = \tau - 1 \wedge (\Gamma_\nu(\tau) \sim \Psi_\nu[X_1(\sigma), \cdots, \Gamma_h(\sigma)])))]\},$$

where $\Gamma_\nu^{\sigma_\nu} = \Gamma_\nu$ for $\sigma_\nu = 1$ and $\bar{\Gamma}_\nu$ for $\sigma_\nu = 0$.

This formula belongs to a one-place predicate calculus, in which only restricted quantifiers for individuals, but unrestricted quantifiers for predicates are admitted. The author proves that, conversely, a formula F of this calculus can be described by a scheme of the form (i), provided it satisfies the following conditions: (a) F contains exactly one free variable t; (b) every quantifier for individuals occurring in F is t-controlled.

The notion of a t-controlled quantifier is defined by the rule: I. $[x]_{x \prec t}$ is t-controlled; II. a quantifier $[\tau]_{\tau<\sigma}$ is t-controlled if every quantifier $[x]_{x\prec \tau}$ occurring in its domain is t-controlled. Here $[x]$ is either (x) or (Ex) and $x \prec t$ is either $x<t$ or $x \leqq t$. *A. Heyting* (Amsterdam)

SET THEORY

5143:

Lorenzen, Paul. Über den Kettensatz der Mengenlehre. Arch. Math. **9** (1958), 1–6.

The chain law referred to in the title is the following result. Let M be a partially ordered set and f an s-compatible mapping of each subset H of M onto an element of M which is an upper bound to H. Then there exists a well-ordered, f-closed subset of M. The basic concept of the chain law is then carried over into operative mathematics in such a way that the existence of a well-ordered f-closed set is not lost. (Notation: f is said to be s-compatible if $f(N_1) = f(N_2)$ whenever $\{x | x \leqq y$ for some $y \in N_1\} = \{x | x \leqq y$ for some $y \in N_2\}$. A set G is said to be f-closed if, for each subset N of G, $N \cup f(N)$ is a subset of G.) *S. Ginsburg* (Hawthorne, Calif.)

5144:

Stupina, I. D. Properties of some δs-operations. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 188–190. (Russian)

The paper is connected with researches of A. A. Lyapunov, Z. I. Kozlova and others on the descriptive set theory [for terminology, notations and references cf. Z. I. Kozlova, Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 349–370; MR **19**, 829; and I. D. Stupina, ibid. 835–862; MR **20** #4486]. The paper consists of seven theorems without proof. E.g., theorems 1 and 2 read, respectively: If N_c' is a rigid basis of a Γ-operation, then the operations

$$\phi_{N;\cdots}, \ \phi_{N;\cdots}$$

are not stronger than the Γ-operation; if N_c' is a rigid basis of a CA_2-operation, then

$$\Gamma_{N;\cdots}, \ \Gamma_{N;\cdots}$$

are not stronger than the CA_2-operation.
Đ. Kurepa (Zagreb)

5145:

Fraïssé, Roland. Sur quelques classifications des relations, basées sur des isomorphismes restreints. III. Comparaison des parentés introduites dans la première partie avec des parentés précédemment étudiées. Publ. Sci. Univ. Alger. Sér. A 3 (1956), 143–159.

Il s'agit de la troisième partie d'un ouvrage couvrant et généralisant ce que l'A. a dit dans sa thèse [Univ. de Paris, 1953; MR **15**, 296]; pour les parties I et II voir même Publ. **2** (1955), 15–60, 273–295 [MR **18**, 139; **19**, 111]. Voici quelques résultats du § 5: La parenté $\overset{2}{\sim}$ est strictement plus forte que $\sim\overset{2}{\circ}\sim$ et n'entraîne pas $\sim\overset{3}{\circ}\sim$. Pour chaque ordinal $3 \leqq n \leqq \omega$, les parentés $\overset{2}{\sim}$ et $\sim\overset{n}{\circ}\sim$ sont incomparables. Le même § contient un tableau faisant voir schématiquement les relations mutuelles des parentés. Voici une notation utile du § 3. $D_{nn'}{}^p$ (M, M', N, N') ou $D_{nn'}{}^p$: Tout $\begin{bmatrix} n \\ p \end{bmatrix}$-isomorphisme de MN vers $M'N'$ est un $\begin{bmatrix} n' \\ p \end{bmatrix}$-isomorphisme de M vers M'; M, N sont multirelations de même base E; de même M', N' en E'; M et M' ont la même signature, de même N et N'. On a $D_{nn}{}^p$. Si $D_{n,n+1}{}^p$, alors $D_{n+1,n+2}{}^p$. Si $D_{0n}{}^p(M, M, N, N)$, $D_{0n}{}^p(M', M', N', N')$ et $MN \underset{p}{\overset{1}{\sim}} M'N'$, alors $D_{0n}{}^p(M, M', N, N')$ et $M \underset{p}{\overset{n}{\sim}} M'$ (§ 3.4). *Đ. Kurepa* (Zagreb)

5146:

Mickle, E. J.; and Rado, T. On covering theorems. Fund. Math. **45** (1958), 325–331.

A generalization is given of A. P. Morse's [Trans. Amer. Math. Soc. **55** (1944), 205–235; MR **5**, 231] covering theorems. It is proved that if binary relations σ, δ on a set X satisfy certain conditions, there is a set $S \subset X$ such that

(1) $x\sigma y$ for no distinct $x, y \in S$; (2) for any $z \in X$, there is an element $x \in S$ such that $z\sigma x$, $z\delta x$. This result is applied, e.g., to prove some propositions of Vitali type for metric spaces. *M. Katětov* (Prague)

5147:

Ionescu, Har. P. Un théorème sur les nombres ordinaux et quelques conséquences. Bul. Inst. Politehn. Bucureşti 18 (1956), no. 3–4, 35–39. (Romanian. Russian and French summaries)

Let α and β be ordinals and use the definition of product, sum, and power of ordinals as in Sierpiński, "Leçons sur les nombres transfinis" [Gauthier-Villars, Paris, 1928]. If α is transfinite and has cardinal number at least the cardinal of β, then α, 2^α, α^β, and β^α have the same cardinal number. *M. M. Day* (Urbana, Ill.)

5148:

Eyraud, Henri. Le théorème de l'ordinal limite (Compléments). Ann. Univ. Lyon. Sect. A (3) **20** (1957), 5–11.

A revision of the author's latest "proof" of the continuum hypothesis [same Ann. **18** (1955), 5–14; **19** (1956), 7–12; MR **18**, 551; **19**, 829], containing statements that are ambiguous, unsupported, or both, and no new ideas. *L. Gillman* (Princeton, N.J.)

COMBINATORIAL ANALYSIS

See also 5450.

5149:

Al-Salam, W. A. On some theorems on permutations. Amer. Math. Monthly **65** (1958), 615–616.

Let $\Phi_k(n)=\sum_{r=0}^{n} \{{}_nP_r\}^k$, where ${}_nP_r=n!/(n-r)!$ and k is a positive integer. In this classroom note the author extends some theorems given by Mullin [same Monthly **64** (1957), 669–670] in the case $k=1$. He shows that a canonic generator for Φ_k is

$$\Phi_k(n+1)=(n+1)^k\Phi_k(n)+1,$$

and then derives a number of asymptotic expressions for $\Phi_k(n)$. *A. L. Whiteman* (Los Angeles, Calif.)

ORDER, LATTICES

See also 5137, 5143, 5154, 5155, 5464, 5465.

5150:

Barbălat, I. Ensembles cofinals. Bul. Inst. Politehn. Bucureşti **18** (1956), no. 3–4, 51–55. (Romanian. Russian and French summaries)

The author is concerned with the possibility of isomorphism between some cofinal subsets of two given cofinal subsets of a partially ordered set E. He achieves some results in two cases: E well ordered, and E a set with at least one maximal element above each element of E. *M. M. Day* (Urbana, Ill.)

5151:

Monteiro, A. A. Les ensembles ordonnés compacts. Rev. Mat. Cuyana **1** (1955), 187–194.

The author proves a generalization of a theorem of J. W. Alexander which states that a topological space is compact if there exists a sub-basis for the closed sets of the space having a certain intersection property. This generalization deals with partially ordered sets rather than topological spaces. A partially ordered set is called semi-orthogonal if it has a zero element, and if, whenever a, b, and c are elements such that $a\neq 0$, $a\frown b=a\frown c=0$, then there exists an element x such that $a\frown x=0$, $b\leq x$, and $c\leq x$. For example, any distributive lattice with a zero element is semi-orthogonal.

The principal theorem states that a semi-orthogonal partially ordered set is compact if and only if it has a compact sub-basis. A set S is called a sub-basis if every element is the greatest lower bound of a set of elements each of which is the least upper bound of a finite set of elements of S. A set S of elements of a partially ordered set P is called compact if whenever a subset K of S has the property that every finite subset of K has a non-zero common predecessor, then K has a non-zero common predecessor. This theorem is non-trivial, and is proved by the use of Zorn's lemma. The theorem of Tychonoff on the compactness of cartesian products of compact spaces is an immediate consequence. *O. Frink, Jr.* (University Park, Pa.)

5152:

Choudhury, A. C. The doubly distributive m-lattice. Bull. Calcutta Math. Soc. **49** (1957), 71–74.

The author discusses "doubly distributive m-lattices", defined as m-lattices, in which $a(b\cup c)=ab\cup ac$ and $(a\cup b)c=ac\cup bc$ (as well as dually). *G. Birkhoff* (Cambridge, Mass.)

5153:

Balachandran, V. K. On isomorphic BS-representations preserving arbitrary joins. Math. Japon. **4** (1956), 55–61.

The author continues his study of BS-representations of lattices [Proc. Indian Acad. Sci. Sect. A **45** (1957), 35–46; MR **20** #13]. Among his new results is the theorem that, among complemented lattices, the atomic Boolean algebras are precisely those which have isomorphic BS-representations preserving arbitrary joins. *G. Birkhoff* (Cambridge, Mass.)

GENERAL ALGEBRAIC SYSTEMS

See also 5121, 5249.

5154:

Mal'cev, A. I. The structure characteristic of some classes of algebras. Dokl. Akad. Nauk SSSR **120** (1958), 29–32. (Russian)

The author seeks conditions on a category of lattices in order that it be lattice-equivalent to a subclass of a class of algebras of a fixed type, a subclass which is multiplicatively closed and which contains subalgebras of its algebras. The notions of a quasi-free subcategory and of rational equivalence are discussed. Other theorems concern a category L of lattices which contains L-free lattices with L-finitarily dense L-free systems of arbitrary power. *R. A. Good* (College Park, Md.)

5155:

Mal'cev, A. I. On certain classes of models. Dokl. Akad. Nauk SSSR **120** (1958), 245–248. (Russian)

In order that a category K be lattice equivalent to a

universally axiomatized class of models, it is necessary and sufficient that K be locally consistent and have finitary homomorphisms, and that any subset of a K-lattice be a K-sublattice of it. The lattice characterization of quasi-primitive classes of algebraic systems is investigated. Suppose a regular category K is multiplicatively closed, contains the identity lattice, is closed homomorphically into itself, and has the property that under a homomorphic mapping of a K-lattice onto a K-lattice the counterimage of a K-sublattice is a K-sublattice; suppose K contains a finite lattice $\mathfrak{A}$; then the following three conclusions hold: K-free lattices with differing finite numbers of free generators are not isomorphic; in the quasi-free closure of the class $\{\mathfrak{A}\}$ and in the free closure of $\{\mathfrak{A}\}$, every lattice with a finite generating set is finite; if the number of nonisomorphic K-lattices of each finite cardinal is finite, then in the free closure of $\{\mathfrak{A}\}$ there are only finitely many non-identity minimal quasi-free and minimal free subclasses.

R. A. Good (College Park, Md.)

5156:

Lazarson, T. The representation problem for independence functions. J. London Math. Soc. 33 (1958), 21–25.

In the present paper the author deals with questions concerning the representability of independence (I-)functions [R. Rado, Proc. London Math. Soc. (3) **7** (1957), 300–320; MR **19**, 522]. Two results in this respect are presented. Let $n=pr+1$, p being a prime, r a positive integer, and let $e_1, \cdots, e_n$ stand for a linearly independent sequence of vectors in a left vector space over a division ring Δ of characteristic p. The author shows that if the natural I-function on the set $\{e_1, \cdots, e_n;\ u-e_1, \cdots, u-e_n, u\}$, where $u=e_1+\cdots+e_n$, can be represented over a division ring D, then the characteristic of D is a factor of pr; in the case of $r=1$, D has characteristic p. The other result states the existence of I-functions on a set of 16 elements which are not representable over any division ring.

O. Borůvka (Brno)

THEORY OF NUMBERS

See also 5140, 5141, 5190, 5202, 5235a–c, 5264, 5376, 5395.

5157:

Erdös, Paul. Some unsolved problems. Michigan Math. J. 4 (1957), 291–300.

This is a collection of unsolved problems in number theory, geometry and analysis, many of them due to the author.

S. Chowla (Boulder, Colo.)

5158:

García, Mariano. New amicable pairs. Scripta Math. 23 (1957), 167–171 (1958).

5159:

Cohen, Eckford. Trigonometric sums in elementary number theory. Amer. Math. Monthly 66 (1959), 105–117.

An arithmetical function $f(n, r)$ is said to be an even function of n (mod r) if $f((n, r), r)=f(n, r)$ for all n, where (n, r) denotes the greatest common divisor of n and r. In a previous paper [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 939–944; MR **17**, 713] the author has studied the class of even functions. The purpose of the present article is to illustrate the theory with some instructive examples at an elementary level. Applications are given to the following functions: (1) the Jordan totient function $J_k(r)$; (2) the extended Ramanujan sum $c^{(k)}(n, r)$; (3) the function $\sigma_s^*(n, r)$, defined to be the number of solutions in x_i, y_i (mod r) of the congruence $n\equiv\sum_{i=0}^{s} x_iy_i$ (mod r); (4) the function $\phi_s^*(n, r)$ defined to be the number of solutions in x_i (mod r) of the congruence $n\equiv\sum_{i=0}^{s} x_i$ (mod r), under the restriction that r and the greatest common divisor of $x_0, \cdots, x_s$ be relatively prime; (5) the extended divisor function $\sigma_s(r)$; (6) the extended Euler function $\phi_s(r)$.

A. L. Whiteman (Los Angeles, Calif.)

5160:

Khan, N. A. On some congruences of idempotent matrices. Proc. Nat. Inst. Sci. India. Part A. 24 (1958), 1–3.

Two matrices of order n, A and B, are said to be congruent (mod n) if their corresponding elements a_{rs} and b_{rs} are congruent (mod n).

The author proves that if A is idempotent and p satisfies $2^p\equiv2$ (mod p), then $(I+A)^p-A\equiv I$ (mod p), $[2(I-A)]^p+2A\equiv2I$ (mod p), $[4(I-A)]^p+4A\equiv4I$ (mod p). (I is the identity matrix.) He also proves that if any of the above congruences hold, then $2^p\equiv2$ (mod p).

P. Erdõs (Haifa)

5161:

Draim, N. A. Test for divisibility by the use of a remainder function. Math. Mag. 31 (1957/58), 137–140.

This paper illustrates by way of examples a numerical procedure for testing a positive integer for divisibility by using a sequence of remainder functions. That the algorithm is an effective one is, of course, of interest. (A method of proof of this is indicated.) It is not, however, an efficient one, as the author fails to point out. If the given number N happens to be a large prime, for example, the algorithm would fail to discover this until at least $N^{\frac{1}{2}}/2$ divisions, even though the successive dividends form a strictly decreasing sequence.

O. Gross (Santa Monica, Calif.)

5162:

Gutnik, L. A. On the arithmetic of matrices. Dokl. Akad. Nauk SSSR 121 (1958), 786–789. (Russian)

Let A, B be non-singular matrices of order n with integral elements. B is called a left divisor of A if $B^{-1}A$ is integral. Two left divisors B_1, B_2 of A are considered as equivalent if $B_2^{-1}B_1$ is integral and unimodular. Clearly, two equivalent left divisors of A have the same invariant factors, but not conversely. For a diagonal matrix $C=\operatorname{diag}(c_{11}, c_{22}, \cdots, c_{nn})$ such that each c_{ii} is a positive integer and $c_{ii}|c_{i+1,i+1}$, the author investigates the number $t(A, C)$ of equivalence classes of those left divisors of A whose invariant factors are the diagonal elements of C. An explicit expression for computing $t(A, C)$ is given. Let P, Q be non-singular, skew-symmetric, integral matrices of order $2n$. Denote by $\Gamma(P, Q)$ the set of all integral matrices X satisfying $X'PX=Q$, where X' is the transpose of X. It is natural to regard two solutions X_1, X_2 of the equation $X'PX=Q$ as left-equivalent [right-equivalent], if $X_2=YX_1$ with $Y\in\Gamma(P, P)$ [if $X_2=X_1Z$ with $Z\in\Gamma(Q, Q)$]. Let $a_l(P, Q)$ [$a_r(P, Q)$] denote the number of left-equivalence classes [right-equivalence classes] of solutions X of $X'PX=Q$. Expressions for $a_l(P, Q)$ and $a_r(P, Q)$ are obtained. Also the following theorem is announced, which is analogous to a classical theorem of Hasse on quadratic forms. Let P, Q be two non-singular, skew-symmetric, integral matrices of order $2n$, and let q be a positive

integer divisible by the cubic power of $\det(PQ)$. Then there exists an integral matrix X satisfying $X'PX=Q$ if and only if the congruence $X'PX\equiv Q \pmod{q}$ is solvable.

Ky Fan (Notre Dame, Ind.)

5163:

Leech, John. Groups of primes having maximum density. Math. Tables Aids Comput. 12 (1958), 144–145.

The author lists all groups of $k\geqq 6$ primes $p_1<p_2<\cdots<p_k$ which minimize the difference p_k-p_1, in the range 50 to 10^7. A list of the groups with $k=4$ and 5 has been deposited in the UMT file of MTAC.

F. A. Behrend (Melbourne)

5164:

Djerasimović, Božidar. Über die binären quadratischen Formen. Math. Z. 66 (1957), 328–340.

This is an exposition of known results on the connection between binary quadratic forms and continued fractions, with slight simplifications resulting from the author's notation for ordered sets of numbers, introduced in two previous papers [same Z. **62** (1955), 320–329; **66** (1956), 228–239; MR **17**, 255; **18**, 635].

H. Davenport (Cambridge, England)

5165:

Anđelić, T. P. Une méthode pratique pour la décomposition de formes quadratiques aux coefficients numériques en sommes de carrés. Glas Srpske Akad. Nauka 232 Od. Prirod.-Mat. Nauka (N.S.) 15 (1958), 45–57. (Serbo-Croatian. French summary)

5166:

Birch, B. J.; and Davenport, H. On a theorem of Davenport and Heilbronn. Acta Math. 100 (1958), 259–279.

Davenport and Heilbronn have proved that if $\lambda_1, \cdots, \lambda_5$ are any real numbers not all of the same sign and none of them zero, there are integers $x_1, \cdots, x_5$ not all zero such that

$$(*)\qquad |\lambda_1 x_1^2+\cdots+\lambda_5 x_5^2|<1.$$

This paper sharpens this result by means of the following theorem. For $\delta>0$ there exists C_δ with the following property. For any real $\lambda_1, \cdots, \lambda_5$, not all of the same sign and all of absolute value 1 at least, there exist integers $x_1, \cdots, x_5$ which satisfy both (*) and

$$0<|\lambda_1|x_1^2+\cdots+|\lambda_5|x_5^2<C_\delta|\lambda_1\cdots\lambda_5|^{1+\delta}.$$

This result is applied in another paper by the same authors [Mathematika **5** (1958), 8–12; MR **20** # 3104] to general indefinite quadratic forms.

B. W. Jones (Boulder, Colo.)

5167:

Birch, B. J. The inhomogeneous minimum of quadratic forms of signature zero. Acta Arith. 4 (1958), 85–98.

Minkowski's theorem on the product of two non-homogeneous linear forms can be stated as follows. Let $Q(x, y)$ be an indefinite binary quadratic form of determinant D, with real coefficients. Let

$$M_I(Q)=\sup\inf Q(x^*+x, y^*+y),$$

where "inf" is over integers x, y and "sup" is over real x^*, y^*. Then $M_I(Q)\leqq|\tfrac14 D|^{\frac12}$. The present paper contains a generalization of this: if Q_{2n} is a quadratic form in $2n$ variables of signature zero, with real coefficients, then $M_I(Q_{2n})\leqq|\tfrac14 D|^{1/(2n)}$. Further, the sign of equality is needed if and only if Q is equivalent to a multiple of the form $x_1x_2+x_3x_4+\cdots+x_{2n-3}x_{2n-2}+2x_{2n-1}x_{2n}$. The result is deduced from auxiliary theorems. Thus it is shown that a more precise result holds if $n\geqq 3$ and Q_{2n} does not represent zero properly; also that $M_I(Q_{2n})=0$ if $n\geqq 2$ and Q_{2n} is not a multiple of a rational form and assumes values arbitrarity near to 0 (including 0). The detailed work of the paper is difficult, though technically elementary, and uses a variety of methods.

H. Davenport (Cambridge, England)

5168:

Vinogradov, I. M. A special case of estimation of trigonometric sums involving prime numbers. Izv. Akad. Nauk SSSR. Ser. Mat. 22 (1958), 3–14. (Russian)

This is a continuation of the author's previous paper [same Izv. **21** (1957), 145–170; MR **19**, 839]. As there, let $f(x)=A_nx^n+\cdots+A_1x$, and suppose that $n\geqq 12$. Let $S=\sum_{p\leq P}\exp(2\pi ikf(p))$, where p runs through primes and $k>0$ is an integer. Let $a_1/q_1, \cdots, a_n/q_n$ be irreducible fractions which approximate to $A_1, \cdots, A_n$ in the sense that $A_s-a_s/q_s=P^{-s}\delta_s$, where $|\delta_s|\leqq P^{1/n}$, for $s=1, \cdots, n$. Let $Q=$l.c.m. of $q_1, \cdots, q_n$. The present paper is concerned with the case $Q\leqq P^{1/n}$. Theorem 1 states that if $k\leqq P^{3/n^2}$, then $S=O(P\exp\{19(\log\log P)^2\}(k, Q)^{1/(2n)}Q^{\varepsilon-1/(2n)}$ for any fixed $\varepsilon>0$. Moreover, if there is a value s' of s for which $|\delta_{s'}|\geqq 1$, then the factor $(k, Q)^{1/(2n)}$ in the above estimate can be replaced by $|\delta_{s'}|^{-1/(2n)}$. Theorem 2 relates to the case when Q is still smaller, namely $Q\leqq\exp(\log P)^{\frac12}$; but the author's final remark indicates that the sum S can then be estimated more effectively by using the theory of primes in arithmetic progressions.

H. Davenport (Cambridge, England)

5169:

Korobov, N. M. Approximate calculation of repeated integrals by number-theoretical methods. Dokl. Akad. Nauk SSSR (N.S.) 115 (1957), 1062–1065. (Russian)

Let $f(x_1, x_2, \cdots, x_s)$ be a function of period 1 in each variable which is expressible in an absolutely convergent s-dimensional Fourier series and let σ be the sum of the moduli of its Fourier coefficients. Let p be a prime greater than s and put $\xi_\nu(k)=\{k^\nu/p^2\}$ (fractional part) for $1\leqq\nu\leqq s$. If the partial derivative

$$\frac{\partial^{2s}f(x_1, x_2, \cdots, x_s)}{\partial x_1^2\partial x_2^2\cdots\partial x_s^2}$$

is continuous and if, for any integers $j_1, j_2, \cdots, j_r$, $1\leqq j_i\leqq s$, $1\leqq r\leqq s$, the partial derivatives $\partial^{2r}f/\partial x_{j_1}^2\partial x_{j_2}^2\cdots\partial x_{j_r}^2$ are bounded in modulus by a constant C, then, for $N=p^2$, the author shows that

$$\left|\int_0^1\cdots\int_0^1 f(x_1, \cdots, x_s)dx_1\cdots dx_s - \frac{1}{N}\sum_{k=1}^{N} f(\xi_1(k), \cdots, \xi_s(k))\right|\leqq\frac{(s-1)\sigma}{\sqrt{N}}+\frac{sC}{10N}.$$

Here the points $(\xi_1(k), \cdots, \xi_s(k))$ depend upon N. The author indicates how this result can be modified so that the dependence of these points on N is lessened.

R. A. Rankin (Glasgow)

5170:

Hua, Loo-keng; and Wu, Fang. An improvement of Vinogradov's mean value theorem and some applications. Acta Math. Sinica 7 (1957), 574–589. (Chinese. English summary)

The authors prove the following improvement of Vinogradov's mean value theorem for trigonometrical sums: Let

$$f(x)=\alpha_kx^k+\alpha_{k-1}x^{k-1}+\cdots+\alpha_1x,\quad C_k(P)=\sum_{n=1}^{P}e^{2\pi if(x)}.$$

There exist two absolute constants A and B such that, if l is a positive integer satisfying $\frac{1}{4}k(k+1)+lk\leqq t\leqq Ak^2\log k$, then

$$\int_0^1\cdots\int_0^1|C_k(P)|^{2l}d\alpha_1\cdots d\alpha_k\leqq e^{Bk^3\log k}(\log P)^l P^{2l-\frac{1}{2}k(k+1)+\delta_l},$$

where $\delta_l=\frac{1}{2}k(k+1)(1-1/k)^l$. From this inequality, they deduce that

$$\pi(x)=\mathrm{li}\,x+o(xe^{-A(\log x)^{4/7}(\log\log x)^{-1/7}})$$

and

$$\zeta(1+it)=o((\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}).$$

{Since the paper was written, Vinogradov announced further improvements of his method at the Edinburgh International Congress, 1958.} *K. Mahler* (Manchester)

5171:

Vinogradov, I. M. On the $\zeta(s)$ function. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 631. (Russian)

The author announces the estimate

$$\zeta(1+it)=O((\log t\log\log t)^{\frac{2}{3}})$$

as $t\to\infty$. This result and others like it lead to a reduction in the estimate for the remainder in the asymptotic formula for $\pi(x)$. Also it follows that $\zeta(s)$ has no zero $\sigma+it$ in the region $\sigma\geqq 1-C(\log t\log\log t)^{-\frac{2}{3}}$ for some constant C.

A detailed account of these results is promised.

D. H. Lehmer (Berkeley, Calif.)

5172:

Mikolás, Miklós. Sur l'expression fermée des séries

$$\sum_{k=1}^{\infty}k^{-(2\nu+1)}\quad(\nu=1,2,\cdots)$$

et le rapport $\zeta(s,u)/\zeta(s)$. Mat. Lapok **8** (1957), 100–107. (Hungarian. French and Russian summaries)

The principal result of this paper is the integral relation

$$\int_0^1[\zeta(1-s,u)-\zeta(1-s)]\cot\pi u\,du=2(2\pi)^{-s}\Gamma(s)\sin\frac{\pi s}{2}\,\zeta(s),$$

where $\zeta(s)$ is Riemann's zeta function, $\zeta(s,u)$ is Hurwitz' zeta function, and Re $s>1$. One of the consequences is

$$\int_0^1\left[\frac{\zeta(s,u)}{\zeta(s)}-1\right]\cot\pi u\,du=\cot\frac{\pi s}{2},$$

valid for Re $s<0$, $s\neq-2,-4,\cdots$.

A. Erdélyi (Pasadena, Calif.)

5173:

Tong, Kwang-Chang. On divisor problems. II, III. Acta Math. Sinica **6** (1956), 139–152, 515–541. (Chinese. English summary)

Let $d_k(n)$ be the number of integral solutions of $n=n_1\cdots n_k$; let $D_k(x)=\sum_{n\leq x}d_k(n)$; let $R_k(x)$ be the residue of $\zeta^k(s)x^s/s$ at $s=1$; and let $\Delta_k(x)=D_k(x)-R_k(x)$. The author continues the work of his earlier paper with the same title [same Acta **5** (1955), 313–324; MR **17**, 462]. He proves in part II that, if l is any positive integer, then

$$\limsup_{x\to\infty}\frac{(-1)^{(k-1)/4}\Delta_k(x)}{(x\log x)^{(k-1)/2k}(\log\log x)^{k-1}}>0\quad\text{if }k\neq 4l+1,$$

$$\liminf_{x\to\infty}\frac{(-1)^{(k-1)/4}\Delta_k(x)(\log\log\log x)^{(k+1)(k-2)/2k}}{x^{(k-1)/2k}(\log x)^{(k-2)/2k}(\log\log x)^{k-2}}<0\quad\text{if }k=2l,$$

$$\liminf_{x\to\infty}\frac{(-1)^{(k-1)/4}\Delta_k(x)(\log\log\log x)^{(k-1)(k-3)/2k}}{x^{(k-1)/2k}(\log x)^{(k-3)/2k}(\log\log x)^{k-2}}<0\quad\text{if }k=4l-1,$$

$$\lim_{x\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\Delta_k(x)(\log\log\log x)^{(k^2-1)/2k}}{(x\log x)^{(k-1)/2k}(\log\log x)^{k-2}}\begin{Bmatrix}>\\ <\end{Bmatrix}0\quad\text{if }k=4l+1.$$

Let, further, σ_k be the lower bound of σ for which

$$\int_{-T}^{+T}|\zeta(\sigma+it)|^{2k}dt\ll T^{1+\varepsilon},$$

where $\varepsilon>0$ is arbitrarily small. The main result of part III is then that

$$\int_0^x\Delta_k(y)^2dy=\frac{1}{(4k-2)\pi^2}\sum_{n=1}^{\infty}\frac{d_k(n)^2}{n^{1+1/k}}x^{2-1/k}+\begin{cases}O(x^{2+\varepsilon-(3-4\sigma_k)/(2k(1-\sigma_k)-1)}) & \text{if }k>2,\\ O(x(\log x)^5) & \text{if }k=2.\end{cases}$$

K. Mahler (Manchester)

5174:

Golubev, V. A. Sur certaines functions multiplicatives et le problème des jumeaux. Mathesis **67** (1958), 11–20.

By considering Schemmel's generalization of Euler's totient function, namely $\varphi_2(n)=n\prod_{p|n}(1-2p^{-1})$, the author gives a heuristic argument in support of the Hardy-Littlewood conjecture that the number $\pi_2(x)$ of twin primes $\leqq x$ is given by $\pi_2(x)=C\{\pi(x)\}^2/x$, where $C=2\prod_{p>3}\{1-(p-1)^{-2}\}=1.32023631$.

The argument is extended to triplets of primes with given differences and so on. Six small tables give data on the numbers of twin primes, triplets, quadruplets and quintuplets below 5000,000, 3,000,000, 2,000,000 and 2,000,000 respectively. {Some of these results are slightly erroneous. For example, the number of primes $p<$ 3,000,000 for which $p+2$ and $p+6$ are also primes is 3273, not 3266, an easily understood error when one considers that the work was done by hand. [On twin primes cf. results of Sexton, Boll. Un. Mat. Ital. (3) **10** (1955), 99–101; MR **16**, 796.] The reader should be on the lookout for misprints such as $\pi(x-\sqrt{x})$ for $\pi(x)-\pi(\sqrt{x})$.} *D. H. Lehmer* (Berkeley, Calif.)

5175:

Bredihin, B. M. Free numerical semigroups with power densities. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 855–857. (Russian)

The author remarks that the elementary proof of the prime number theorem utilizes only the multiplicative and order properties of the integers and hence the method is available for asymptotic density theorems for multiplicative semi-groups of real numbers. Let $1=\alpha_1<\alpha_2<\alpha_3<\cdots$ be generated multiplicatively by a set of generators $1<\omega_1<\omega_2<\omega_3<\cdots$ so that each α is uniquely a product of powers of a finite number of ω's. Let $\nu(x)$ and $\pi(x)$ be the number of α's and ω's, respectively, not exceeding x. It is assumed that, for some $\theta>0$, $x^{-\theta}\nu(x)$ tends to a positive limit as $x\to\infty$. Then it is proved by elementary methods that

$$\pi(x)\sim y/\log y\quad(y=x^\theta).$$

Besides its application to Gaussian primes and prime ideals the author gives an application to the following "inverse" problem. Suppose that the number of solutions $(n_1,n_2,\cdots)$ in positive integers of the inequality $n_1\alpha_1+n_2\alpha_2+\cdots\leqq x$ is $Ce^{\theta x}+O(e^{\theta_1 x})$ for some $\theta>\theta_1>0$. Then $\nu(x)=y/\log y$ $(y=e^{\theta x})$. *D. H. Lehmer* (Berkeley, Calif.)

5176:

✱Ostmann, Hans-Heinrich. Additive Zahlentheorie. Erster Teil: Allgemeine Untersuchungen. Zweiter Teil: Spezielle Zahlenmengen. Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Hefte 7, 11. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. vii+233 pp. DM 29.80. vi+136 pp. DM 22.00.

These two volumes constitute a very comprehensive

survey of the subject with particular emphasis placed on the theory of density for sets of non-negative integers. Some idea of the magnitude of the work can be gleaned from the fact that the text cites about 800 papers and books (whereas Landau's monumental Handbuch der Lehre von der Verteilung der Primzahlen [Chelsea Publ. Co., New York, 1953; MR **16**, 904], of three times the size, mentions about 650). We also note that with the exception of Landau's Über einige neuere Fortschritte der additiven Zahlentheorie [Cambridge Univ. Press, London, 1937] this is the only book to give a connected account of the theory of density; moreover, it does this in a much more general and extensive fashion than the earlier book.

It is difficult to give an idea of the detailed results proved or stated in the work. Roughly speaking, the problem is to deduce properties of certain sets formed from given sets $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ of nonnegative integers if certain properties are assumed for the given sets $\mathfrak{A}_k$. Among the sets which can be formed from the $\mathfrak{A}_k$ are the set-theoretic union and intersection and the set $\sum_{k=1}^{n} \mathfrak{A}_k$, the totality of all possible sums $\sum_{k=1}^{n} a_k$ with $a_k \in \mathfrak{A}_k$. One can also form the sets $h\mathfrak{A}$, $h\times\mathfrak{A}$ and $\mathfrak{A}_{\mathfrak{B}}$, whose definitions we omit. Moreover, it is possible to define various relations among the sets and to introduce a non-Archimedean metric on the space of all sets. Among the various properties of sets that are treated can be found the following: rationality, pseudo-rationality, transcendence, finite bases, asymptotic bases, minimal bases, primitiveness, $\mathfrak{T}$-free, relative zeros and essential components. Furthermore, there are many properties which can be stated in terms of certain real-valued functions defined on a set or a collection of sets; among such functions are the counting function, the Fermat index, the asymptotic Fermat index, the various densities as well as the number of compositions and partitions of an integer n when the summands are selected from preassigned sets and are subject to certain auxiliary restrictions. As for the densities, here we find the whole catalog: arithmetic (Schnirelmann), asymptotic, Dirichlet, logarithmic, modified Schnirelmann, natural, n-termed, upper asymptotic, varied, and all the corresponding Stohr $\phi(x)$ densities. Various inequalities connecting the different densities for the same sets are explored in great detail as are the inequalities for the same density function applied to different sets; these investigations occupy a large amount of space.

However, other subjects are also dealt with. For example, the following results are proved; Rademacher's series for the unrestricted partition function $p(n)$, identities for partition functions, the Ikehara Tauberian theorem and the Wiener proof of the prime number theorem. And there are also brief discussions of Petersson's work on modular forms, the work of Kummer and others on the Fermat problem, as well as more extended summaries of work on the Goldbach, Waring and similar problems.

Here, then, is a veritable wealth of proof or statement compressed into about 300 pages of text. This compression has been accomplished with the aid of a highly concise notation including a rather liberal use of logical connectives and symbols set aside, once for all, for special sets. This frequently results in a rather forbidding appearance. Unfortunately, also, the list of abbreviations and the index are not equal to the task of enabling one to quickly find the meaning of symbols defined on previous pages. In addition, almost inevitably, the fullness of coverage makes it difficult to distinguish the more significant results from the lesser ones, but one has the feeling that a better organization of the material might have helped.

But these are small objections to what is certainly a most useful book, one which can serve both as a guide to the literature in the whole field and as a treatise on the theory of density. *L. Schoenfeld* (University Park, Md.)

5177:

Dutta, Mahadev. On new partitions of numbers. Bull. Calcutta Math. Soc. **49** (1957), 221–224.

The author obtains various congruence properties and recurrence relations for partitions whose summands are restricted in several ways; e.g., partitions into m parts with at most d repetitions. *P. Erdős* (Haifa)

5178:

Wintner, Aurel. On the λ-variant of Mertens' μ-hypothesis. Amer. J. Math. **80** (1958), 639–642.

The author considers the conjecture $\sum_{n=1}^{x} \lambda(n) = O(\sqrt{x})$ $(\lambda(n)=(-1)^{\alpha+\beta+\cdots},\ n=p_1^{\alpha}p_2^{\beta}\cdots)$ and proves that the above conjecture is equivalent to $\sum_{n=1}^{x}(\lambda(n)/\sqrt{n}) = c\log x+O(1)$. *P. Erdős* (Haifa)

5179:

Rieger, G. J. Verallgemeinerung der Selbergschen Formel auf Idealklassen mod f in algebraischen Zahlkörpern. Math. Z. **69** (1958), 183–194.

The author extends Selberg's celebrated formula to classes of ideals mod f of an algebraic number field. In fact, he proves that

$$\sum_{\substack{N\mathfrak{p}\le x\\ \mathfrak{p}\in\mathfrak{H}\bmod f}} \log^2 N\mathfrak{p} + \sum_{\substack{N\mathfrak{p}\mathfrak{q}\le x\\ \mathfrak{p}\mathfrak{q}\in\mathfrak{H}\bmod f}} \log N\mathfrak{p}\log N\mathfrak{q} = \frac{2}{h(f)}\, x\log x+O(x);$$

here $\mathfrak{H}$ is any class of ideals mod f and $h(f)$ is the number of the classes of ideals (mod f). The method is similar to that of H. Shapiro [Comm. Pure Appl. Math. **2** (1949), 309–323; Ann. of Math. (2) **52** (1950), 231–243; MR **11**, 501; **12**, 81]. *P. Erdős* (Haifa)

5180:

Rieger, G. J. Ein weiterer Beweis der Selbergschen Formel für Idealklassen mod $\mathfrak{f}$ in algebraischen Zahlkörpern. Math. Ann. **134** (1958), 403–407.

The Selberg formula for arithmetic progressions

$$(1)\quad A_{l,k}(x) = \sum_{\substack{p\le x\\ p\equiv l \bmod k}} \log^2 p + \sum_{\substack{pq\le x\\ pq\equiv l\,(\bmod k)}} \log p\log q = \frac{2}{\varphi(k)}\, x\log x+O(x)$$

follows immediately from the ordinary Selberg formula

$$(2)\quad A(x) = \sum_{p\le x}\log^2 p + \sum_{pq\le x}\log p\log q = 2x\log x+O(x)$$

by means of the relation

$$(3)\qquad A(x) = \frac{1}{\varphi(k)}\, A_{l,k}(x)+O(x).$$

[See A. Selberg, Ann. of Math. (2) **50** (1949), 297–304; MR **10**, 595.] Here, p and q are primes and $(k, l)=1$.

By proving a result analogous to (3) for an arbitrary algebraic numberfield K of degree n (Hilfssatz 8) and using the extension of (2) to algebraic number-fields [H. N. Shapiro, Comm. Pure Appl. Math. **2** (1949), 309–323;

MR 11, 501], namely,

$$(2)'\ \sum_{\mathfrak{N}\mathfrak{p}\leq x}\log^2\mathfrak{N}\mathfrak{p}+\sum_{\mathfrak{N}\mathfrak{p}\mathfrak{q}\leq x}\log\mathfrak{N}\mathfrak{p}\cdot\log\mathfrak{N}\mathfrak{p}=2x\log x+O(x),$$

the author obtains a further proof of the "Selberg formula for ideal-classes modulo $\mathfrak{f}$ in algebraic number-fields" [see W. Forman and H. N. Shapiro, Comm. Pure Appl. Math. 7 (1954), 587–619; MR 16, 114, and #5179 above], namely,

$$(1)'\ \sum_{\substack{\mathfrak{N}\mathfrak{p}\leq x\\ \mathfrak{p}\in\mathfrak{K}\ \mathrm{mod}\ \mathfrak{f}}}\log^2\mathfrak{N}\mathfrak{p}=\sum_{\substack{\mathfrak{N}\mathfrak{p}\mathfrak{q}\leq x\\ \mathfrak{p}\mathfrak{q}\in\mathfrak{H}\ \mathrm{mod}\ \mathfrak{f}}}\log\mathfrak{N}\mathfrak{p}\cdot\log\mathfrak{N}\mathfrak{q}=\frac{2}{h(\mathfrak{f})}x\log x+O(x).$$

Here, $\mathfrak{f}$, $\mathfrak{p}$ and $\mathfrak{q}$ are ideals in K, $\mathfrak{p}$ and $\mathfrak{q}$ being prime ideals. $\mathfrak{H}$ is an ideal-class modulo $\mathfrak{f}$ and $h(\mathfrak{f})$ is the number of classes of ideals prime to $\mathfrak{f}$ modulo $\mathfrak{f}$.

The analogue of (3) is obtained by evaluating

$$F(\mathfrak{a},x)=\sum_{\mathfrak{b}/\mathfrak{a}}\mu(\mathfrak{d})\log^2\frac{x}{\mathfrak{N}\mathfrak{d}}$$

and comparing two different evaluations of the sum

$$\sum_{\substack{\mathfrak{N}\mathfrak{a}\leq x\\ \mathfrak{a}\in\mathfrak{H}\ \mathrm{mod}\ \mathfrak{f}}}F(\mathfrak{a},x).$$

R. A. Rankin (Glasgow)

5181:

Müller, Max. Über die Anzahl der Potenzreste und eine von Herrn Ore angeschnittene Frage. Arch. Math. 9 (1958), 321–341.

The problem in question was the following [Amer. Math. Monthly 63 (1956), 729]: In the number system with base 12 the square numbers end in squares 0, 1, 4, 9. Find all number systems with this property. The author generalizes the problem to determine all basis numbers g for which the last l digits of an nth power always form an nth power. A complete solution is obtained through a detailed evaluation and estimation of the number of nth powers for any given number as modulus. *O. Ore* (New Haven, Conn.)

5182:

Tannaka, Tadao. A generalized principal ideal theorem and a proof of a conjecture of Deuring. Ann. of Math. (2) 67 (1958), 574–589.

Let K be the ray class field mod $\mathfrak{f}$ of the algebraic number field k. Let $\mathfrak{D}(K/k)$ be its different and $\mathfrak{F}(K/k)$ its module of genus (Geschlechtermodul); if $K\supset\Omega\supset k$, let $\mathfrak{f}(K/\Omega/k)=\mathfrak{D}(K/\Omega)\mathfrak{F}(K/k)$. The main result is: Theorem 2. For every k-ideal $\mathfrak{a}$ prime to $\mathfrak{f}$ there exists an element $\Theta(\mathfrak{a})$ with properties: (1) $\mathfrak{a}$, considered as K-ideal, is the principal ideal generated by $\Theta(\mathfrak{a})$ and $\Theta(\mathfrak{a})\equiv1$ mod $\mathfrak{F}(K/k)$; (2) let $\sigma(\mathfrak{a})=(K/k/\mathfrak{a})$ be the Artin automorphism and let $\varepsilon(\mathfrak{a},\mathfrak{b})=\Theta(\mathfrak{a})\Theta(\mathfrak{b})^{\sigma(\mathfrak{a})}/\Theta(\mathfrak{a}\mathfrak{b})$. Then $\varepsilon(\mathfrak{a},\mathfrak{b})$ is an element of the ground field k which is congruent to 1 mod $\mathfrak{f}$, and $\varepsilon(\mathfrak{a},\mathfrak{b})=\varepsilon(\mathfrak{b},\mathfrak{a})$. (Deuring conjectured the special case when K is absolute class field over k.) This is proved by using: Theorem 3. Let K be the direct composition of the cyclic intermediate fields K_i, $i=1,\cdots,r$. For each i, let $\mathfrak{p}_i$ be a prime ideal of k which remains prime in K_i and is fully decomposed in every K_j, $i\neq j$. Let $\sigma_i=(K/k/\mathfrak{p}_i)$. Then there exist K-ideals $\mathfrak{a}_i$ such that $\mathfrak{p}_i\sim\mathfrak{a}_i^{1-\sigma_i}$ mod $\mathfrak{f}(K/K_i/k)$; and for every such system of ideals $\mathfrak{a}_i$ we have $\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_r\sim1$ mod $\mathfrak{F}(K/k)$ in K.

Cohomological application: Let K be the absolute class field over k, G its Galois group, and E its group of units. Let $\mathfrak{a}$ be a k-ideal, τ the Artin-automorphism $(K/\mathfrak{a})$, and define the function E_τ by setting $E_\tau(\sigma)=\Theta(\mathfrak{a})^{1-\sigma}$. Then the mapping $\tau\to E_\tau$ defines an isomorphism of G onto the cohomology group $H^1(G,E)$. Moreover, the function $\tau\to E_\tau(\sigma)$ is a cocycle for fixed σ : $E_\lambda(\sigma)E_\mu{}^\lambda(\sigma)=E_{\lambda\mu}(\sigma)$.

G. Whaples (Bloomington, Ind.)

5183:

Igusa, Jun-ichi. Class number of a definite quaternion with prime discriminant. Proc. Nat. Acad. Sci. U.S.A. 44 (1958), 312–314.

M. Eichler [Math. Z. 43 (1938), 102–109] showed that the class number for a definite quaternion algebra of prime discriminant, ramified only at p, is

$$h=(1/3)(1-(-3/p))+(1/4)(1-(-4/p)+(1/12)(p-1).$$

M. Deuring [Abh. Math. Sem. Hansischen Univ. 14 (1941), 197–272; MR 3, 104] observed that this class number is the number of birationally distinct elliptic curves of characteristic p having no points of order p. The author computes this number directly, getting a purely algebraic proof of the class number formula. The crucial point is that the Hasse invariant of such an elliptic curve (when $p\neq2$) satisfies a differential equation of Gauss-Legendre type. *G. Whaples* (Bloomington, Ind.)

5184:

Newman, Morris. Congruences for the coefficients of modular forms and for the coefficients of $j(\tau)$. Proc. Amer. Math. Soc. 9 (1958), 609–612.

Let $j(\tau)=12^3J(\tau)=x^{-1}+\sum_{n=0}^{\infty}c(n)x^n$ be the absolute modular invariant. The author establishes certain congrences for $c(n)$ to the modulus 13, in particular:

$$c(13n)\equiv-\tau(n),$$

$$c(13np)+c(13n)c(13p)+p^{11}c(13n/p)\equiv0,$$

$$c(13np^{2a-1})\equiv0,$$

all modulo 13. In the last congruence, $13|\tau(p)$ and $(n,p)=1$ (for example, $p=7, 11, 157, 179$). Here n and a are positive integers, $\tau(n)$ is Ramanujan's function, and $c(u)=0$ if u is not an integer.

These results extend previously obtained congruences by Lehmer [Amer. J. Math. 64 (1942), 488–502; MR 3, 272], Lehner, [ibid. 71 (1949), 136–148, 373–386; MR 10, 357], and van Wijngaarden [Indag. Math. 15 (1953), 389–400; MR 15, 403].

J. Lehner (East Lansing, Mich.)

5185:

Varnavides, P. Antisymmetric Markoff forms. Nederl. Akad. Wetensch. Proc. Ser. A 61=Indag. Math. 20 (1958), 463–469; erratum, 62 (1959), 328.

Markov forms are integral indefinite forms

$$f(x,y)=mx^2+pxy+qy^2,$$

with homogeneous minimum m and discriminant $d=9m^2-4$. Here m can be an arbitrary positive integer for which the diophantine equation $m^2+y^2+z^2=3myz$ is soluble. The present author uses Markov's continued fraction machinery and constructs two sets of forms of the above type, with $d=9m^2+4$; the numbers m are those for which $m^2+y^2-z^2=3myz$ has a solution with $z=1$ and $z=2$, respectively. *C. G. Lekkerkerker* (Amsterdam)

5186:

Šalát, Tibor. Zu einer Eigenschaft der Irrationalzahlen. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 7 (1957), 128–137. (Slovak. Russian and German summaries)

Two players alternately choose positive real numbers

$a_1, a_3, a_5, \cdots$ and $a_2, a_4, \cdots$, in such a way that for every n, $a_{n+1}<a_n$. It is shown that either player can force the following outcome: $\prod_1^\infty (1+a_n)$ converges to an irrational number. This is in analogy with a result due to A. Turowicz [Ann. Polon. Math. **2** (1955), 103–105; MR **17**, 466] concerning $\sum_1^\infty a_n$. *W. J. LeVeque* (Göttingen)

5187:

Erdös, Paul. Sur certaines séries à valeur irrationnelle. Enseignement Math. (2) **4** (1958), 93–100.

A. Oppenheim asked if

$$\sum_{n=1}^{\infty} \frac{p_n{}^k}{n!} \quad (k=1, 2, 3, \cdots)$$

is irrational. Here p_n is the nth prime.

The author proves this for $k=1$, and states that the proof is more complicated when $k>1$. When $k=1$ he uses $p_{n+1}-p_n=o(p_n)$. He also raises the problem: Is $\sum_1^\infty p_n/2^n$ irrational? *S. Chowla* (Boulder, Colo.)

5188:

Ehrhart, Eugène. Sur les inéquations diophantiennes linéaires à deux inconnues. C. R. Acad. Sci. Paris **246** (1958), 2987–2989.

Let a and b be relatively prime positive integers. The author studies the number n_C of pairs of positive integers X, Y satisfying $aX+bY<C$. In particular, he shows that if $C=abq+r$, where q, r are integers with $0<r<q$, then

$$n_C=n_r+\tfrac{1}{2}q(C+r-a-b-1).$$

C. A. Rogers (Birmingham)

5189:

Ehrhart, Eugène. Nombre de solutions de l'équation et de l'inéquation diophantiennes linéaires à trois inconnues. C. R. Acad. Sci. Paris **246** (1958), 3142–3145.

Let a, b, c be positive integers, relatively prime in pairs. The author studies the numbers n_D and N_D of triples of positive integers X, Y, Z satisfying $aX+bY+cZ=D$ and $aX+bY+cZ<D$. In particular, he shows that if $D=abcq+r$, where q, r are integers with $0<r<q$, then

$$n_D=n_r+\tfrac{1}{2}q(D+r-a-b-c),$$

$$N_D=N_{abcq}+N_r+1+\tfrac{1}{2}qr(D-a-b-c-1),$$

$$N_D=qN_{abc}+N_r+q\{\tfrac{1}{6}(q^2-1)a^2b^2c^2 -\tfrac{1}{4}(q-1)abc(a+b+c+1)+\tfrac{1}{2}r(D-a-b-c-1)+1\}.$$

C. A. Rogers (Birmingham)

COMMUTATIVE RINGS AND ALGEBRAS

See also 5218, 5221.

5190:

✱Albert, A. Adrian. Fundamental concepts of higher algebra. The University of Chicago Press, Chicago, Ill., 1958. ix+165 pp. $6.50.

"We give here a compact and self-contained exposition of the fundamental concepts of modern algebra which are needed for a clear understanding of the place of finite field theory in modern mathematics. ⋯ The real reason for the book is Chapter 5 in which we present a modern and improved exposition of the theory of finite fields." (From the preface)

Chapter headings: (1) Groups; (2) Rings and fields; (3) Vector spaces and matrices; (4) Theory of algebraic extensions; (5) Finite fields. An appendix contains tables of primitive roots and irreducible polynomials.

O. Ore (New Haven, Conn.)

5191:

Racine, C. Contribution to the Galois theory. J. Madras Univ. Sect. B. **26** (1956), 643–647.

This is a skeletal exposition of some theorems of Galois theory. The contribution of the title is identified by the author as the use of the trace function in the proof of one of these. Apparently he has been anticipated, since this device may be found in the proof of the same theorem on page 94 of E. Artin's "Modern higher algebra, Galois theory lectures given in the summer of 1947" [Notes by Albert Blank, Institute for Mathematics, New York University, 1957 (mimeographed)].

{Reviewer's notes: (1) Theorem II will be incorrect unless the usual exclusion of the zero mapping in the definition of a "character" is observed. (2) "Normal extensions" are characterized in a theorem but are not defined in the text. In context these are the "extensions normales" of one of the bibliographical references. (3) The assumption in theorem IV that E is a "normal extension" (="extension normale") of a subfield K of F seems not used in the author's proof. It is, of course, not needed.} *C. C. Faith* (University Park, Pa.)

5192:

Aurora, Silvio. On power multiplicative norms. Amer. J. Math. **80** (1958), 879–894.

Amongst other things, this paper extends results of A. Ostrowski [Acta Math. **41** (1917), 271–284] from the case of valued fields to that of rings with pseudo- (=semi-) norms satisfying one or more conditions imposed en route. This extension culminates from more general considerations. The set of pseudonorms N carried by a ring R forms a semi-lattice under the obvious partial ordering: the upper envelope of a bounded family of pseudonorms is again a pseudonorm, termed the join of the family. Special attention is focussed on the power multiplicative (PM) pseudonorms (characterised by the identity $N(x^2)=N(x)^2$) which are stable in the sense that $N(\cdots xy\cdots)=N(\cdots yx\cdots)$. The fundamental theorem asserts that any stable PM pseudonorm $N\neq 0$ is the join of a bounded family of pseudo absolute values (multiplicative pseudonorms). Criteria are given in order that a stable PM pseudonorm shall be archimedean (Theorems 3, 4, 7). Combining the results of this paper with those of an earlier one [Pacific J. Math. **7** (1957), 1279–1304; MR **19**, 1186], the author derives the Ostrowski-type theorems for certain categories of rings with a stable PM pseudonorm, the prototype rings being the real numbers, or the real quaternions. The paper ends with a list of all non-zero PM pseudonorms on the rational field or on the ring of integers. *R. E. Edwards* (Woking)

5193:

Samuel, Pierre. Formules de reduction pour les traces et normes. Bol. Soc. Mat. Mexicana **2** (1957), 54–56.

Suppose that R is a Dedekind ring with quotient field K and R' its integral closure in a finite separable extension K' of K. Let $pR'=\prod P_i^{e_i}$ with the residue class homomorphisms h_i: $R'\to R'/P_i=k_i$ and h: $R\to R/p=k$. The author notes that the formula $h(\mathrm{Tr}_{K'/K}(x))=\sum_i e_i \mathrm{Tr}_{k_i/k}(h_i(x)), x\in K'$, for the trace (correspondingly for the norm) is equivalent to the customary formula relating $\mathrm{Tr}_{K'/K}(x)$ with the traces of the images of x in the P_i-

adic completions of K'. Replacing the arguments leading to the proof of the latter, i.e., those of representation theory and completion of vector spaces, he gives a proof which does not require the passage to P_i-adic completion. His method consists of a rather elaborate (necessitated by this type of proof) count and examination of the conjugates lying over the P_i in the least normal closure of K'/K. Essential for the proof is the formula $\sum e_i[h_iR':hR] =[K':K]$. *O. F. G. Schilling* (Chicago, Ill.)

5194:

Northcott, D. G. A note on polynomial rings. J. London Math. Soc. **33** (1958), 36–39.

Using the recent result that if A is a regular local ring with a prime ideal q then A_q is also regular [Auslander and Buchsbaum, Trans. Amer. Math. Soc. **85** (1957), 390–405; MR **19**, 249; Serre, Proc. Internat. Symposium on Algebraic Number Theory, Tokyo and Nikko, 1955, Science Council of Japan, Tokyo, 1956, pp. 175–189; MR **19**, 119], the author's main aim is to prove the following. Let R be a commutative ring with unit, P a prime ideal in $K=R[X_1, X_2, \cdots, X_n]$, and $p=R\cap P$. Then the following two statements are equivalent: (1) K_P is regular; (2) R_p is regular. This generalizes the case where R is a field [Zariski, Trans. Amer. Math. Soc. **53** (1943), 490–542; MR **5**, 11]. It seems appropriate to remark that the implication (2)→(1) is also an immediate consequence of the above Auslander-Buchsbaum-Serre result, their characterization of regular local rings as those of finite global dimension, and gl. dim $R_p[X_1, X_2, \cdots, X_n]=n+$gl. dim R_p [see #5229 below].

A. Rosenberg (Evanston, Ill.)

5195:

af Hällström, Gunnar. Über Halbvertauschbarkeit zwischen linearen und allgemeineren rationalen Funktionen. Math. Japon. **4** (1957), 107–112.

As in a previous paper [Acta Acad. Abo. **21** (1957), no. 2; MR **18**, 887], the author calls two functions $h(z)$ and $g(z)$ semipermutable if there exists a fractional linear function $L(z)=(Az+B)(Cz+D)^{-1}$ $(AD-BC\neq 0)$ such that $h(g(z))=L(g(h(z)))$. In the present note the following problem is solved explicitly: given a fractional linear function $g(z)$, find the most general rational function $h(z)$ which is semipermutable with $g(z)$.

W. Ledermann (Manchester)

ALGEBRAIC GEOMETRY

See also 5183, 5248, 5712.

5196:

Burniat, Pol. Quelques théorèmes d'existence à propos des surfaces algébriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 101–106.

[Pour la première Note, voir même Bull. **44** (1958), 43–55; MR **20**#1676]. L'auteur dans cette deuxième Note se sert de la technique des surfaces quadruples abéliennes de la première Note pour démontrer l'existence de surfaces algébriques régulières dotées des genres géométrique et linéaire suivants: $p_g=1$ et $p^{(1)}=4, 5, \cdots, 9$; $p_g=2$ et $p^{(1)}=4, 5, \cdots, 12$; $p_g=3$ et $p^{(1)}=4, 5, \cdots, 25$, dont le système canonique est irréductible.

M. Piazzolla-Beloch (Ferrara)

5197:

Burniat, Pol. Quelques théorèmes d'existence à propos des surfaces algébriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 230–235.

Dans cette troisième Note l'auteur étend les résultats obtenus dans la Note précédente et les précise. Il peut affermer ainsi l'existence de surfaces algébriques régulières des genres géométrique et linéaire: $p_g=1$, $p^{(1)}=4, 5, \cdots, 9, 11, \cdots, 15$; $p_g=3$, $p^{(1)}=4, 5, \cdots, 30, 31$, dont le système bicanonique ou tricanonique est irréductible et simple. *M. Piazzolla-Beloch* (Ferrara)

5198:

Burniat, Pol. Quelques théorèmes d'existence à propos des surfaces algébriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 327–331.

Dans cette quatrième Note, l'Auteur, élargissant les résultats des précédentes Notes, démontre l'existence de surfaces algébriques régulières possédant les genres: $p_g=2$, et $p^{(1)}=13, 14, \cdots, 17, 19, 20, \cdots, 23$, dont le système bicanonique est simple et irréductible.

M. Piazzolla-Beloch (Ferrara)

5199:

Buquet, A. Sur la détermination de points rationnels d'une cubique à partir de points rationnels de base. Mathesis **67** (1958), 27–44.

If $A_0, \cdots, A_p$ are rational points of a plane elliptic cubic with rational coefficients, Poincaré's formula: $(3n_0+1)A_0+n_1(A_1-A_0)+\cdots+n_p(A_p-A_0)$, where the n_i are rational integers, gives rise to new rational points of the cubic. The author gives an elementary, but very long, rule for the calculation of the rational points given by the above formula. *P. Abellanas* (Madrid)

5200:

Medek, Václav. Einige lineare Systeme von singulären Kollineationen. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 83–93. (Slovak. Russian and German summaries)

Soient x_{ij} les coordonées homogène (dans S_8) de l'homographie $\rho Y^i=\sum_{j=0}^{2} x_{ij}y^j$ d'un plan projectif. La variété VCS_8 qui représente les homographies avec rang $\|x_{ij}\|=1$ se compose de deux systèmes à deux paramètres de plans et coincide avec l'ensemble des points doubles de l'hypersurface V_7^3 qui représente les homographies singulières avec det $|x_{ij}|=0$. L'A. étudie la configuration des plans de la variété V et la représentation sur V des systèmes linéaires $\rho Y^i=\sum_{k,j}\lambda_k x_{ij}^{(k)}y^j$ des homographies singulières. *A. Švec* (Prague)

5201:

Seidenberg, A. Comments on Lefschetz's principle. Amer. Math. Monthly **65** (1958), 685–690.

The author here intends to criticise the so-called principle of Lefschetz that "there is but one algebraic geometry of characteristic p". It seems to the reviewer that modern algebraic geometry has two aspects, which the reviewer will call temporarily "absolute algebraic geometry" and "relative algebraic geometry". Absolute algebraic geometry deals with purely geometric questions, and relative algebraic geometry deals with questions of rationalities of various data. It is the reviewer's understanding that the principle of Lefschetz is meant for absolute algebraic geometry and not for relative algebraic geometry. Essentially, the author shows here that under a suitable condition the principle can be verified for some problems from relative algebraic geometry.

T. Matsusaka (Evanston, Ill.)

5202:

Mattuck, Arthur; and Tate, John. On the inequality of Castelnuovo-Severi. Abh. Math. Sem. Univ. Hamburg 22 (1958), 295–299.

Die in Rede stehende Ungleichung von Castelnuovo und Severi ist diejenige, welche dem Beweis der Riemannschen Vermutung für Kurven zugrundeliegt [vgl. A. Weil, Publ. Inst. Math. Univ. Strasbourg **7** (1945); MR **10**, 262; P. Roquette, J. Reine Angew. Math. **191** (1953), 199–252; MR **15**, 203]. Sie bezieht sich auf Divisoren D einer Fläche der Form $V=C\times C'$, Produkt zweier vollständiger, singularitätenfreier Kurven C und C'. Sie lautet [vgl. Severi, *Trattato di geometrica algebra*, Bologna, 1926]:

$$\text{(1)}\qquad \tfrac{1}{2}[D\cdot D]\leqq dd';$$

dabei sind $d=[D\cdot(P\times C')]$ und $d'=[D\cdot(C\times P')]$ die Grade von D über C bezw. C'. Verff. zeigen nun, daß (1) direkt aus dem Riemann-Rochschen Satz für V hergeleitet werden kann. Dieser läßt sich zufolge der besonderen Struktur von V als Produkt zweier Kurven in der Form

$$\text{(2)}\qquad \chi(D)=\tfrac{1}{2}[D\cdot D]-dd'+(d+1-g')(d'+1-g)$$

schreiben; dabei sind g, g' die Geschlechter von C bezw. C', und $\chi(D)$ bedeutet wie üblich die Eulersche Charakteristik von D, also die alternierende Summe $\chi(D)=\dim D-\sup D+\dim(K-D)$ (K ein kanonischer Divisor auf V). Ihrer Natur nach sind die in $\chi(D)$ eingehenden Terme nichtnegativ; daher ergibt sich die Ungleichung (1) aus der Tatsache, daß $dd'-\frac{1}{2}[D\cdot D]$ als "superabundance" sup E eines geeigneten Divisors E dargestellt werden kann. Und zwar leistet (bei hinreichend großen d, d') der Divisor $E=D-\sum_{i=1}^{d'}(C\times P_i')$ das Verlangte, falls nur die Punkte P_i' auf C' in hinreichend allgemeiner Lage sind. Der Nachweis, daß $\sup E=dd'-\frac{1}{2}[D\cdot D]$, geschieht mit Hilfe von (2), angewandt auf E.

Sind φ_1, φ_2: $C'\to C$ zwei rationale Abbildungen, und sind D_1, D_2 die zugehörigen Graphen, so ergibt sich aus (1), angewandt auf die Divisoren der Form $n_1D_1+n_2D_2$, die Formel

$$\text{(3)}\qquad |\mathrm{Grad}\,\varphi_1+\mathrm{Grad}\,\varphi_2-N|\leqq 2g(\mathrm{Grad}\,\varphi_1\cdot\mathrm{Grad}\,\varphi_2)^{\frac{1}{2}},$$

wobei $N=[D_1\cdot D_2]$ die Anzahl der Lösungen P' von $\varphi_1(P')=\varphi_2(P')$ bedeutet, gezählt mit den Multiplizitäten der algebraischen Geometrie. Diese Formel (3) enthält als Spezialfall die Riemannsche Vermutung für eine Kurve C über einem endlichen Körper k mit q Elementen: dazu hat man $C'=C$ zu wählen, φ_1 als die identische Abbildung, und φ_2 als diejenige Abbildung, welche alle Koordinaten mit q potenziert.

Literatur: zum Riemann-Rochschen Satz für Flächen: Zariski, Ann. of Math. (2) **55** 1952), 552–592 [MR **14**, 180]; Bull. Amer. Math. Soc. **62** (1956), 117–143. Vgl. auch die Arbeit von Grothendieck: J. Reine Angew. Math. **200** (1958), 208–215.
P. Roquette (Hamburg)

5203:

Nakai, Yoshikazu. Some results in the theory of the differential forms of the first kind on algebraic varieties. II. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 31 (1958), 87–93.

[Voir la première partie, Proc. Internat. Sympos. Algebraic Number Theory, Tokyo and Nikko, 1955, pp. 155–173, Science Council of Japan, Tokyo, 1956; MR **18**, 600.] Soit V une hypersurface projective de dimension n; notons s la codimension de l'ensemble S des points singuliers de V (on pose $s=n+1$ si $S=\emptyset$). Si $s>2$, V n'admet aucune forme différentielle de degré $\leqq s-2$ qui soit finie en tout point simple de V (par exemple, une surface nonsingulière de P_3 n'a pas de forme différentielle de première espèce et de degré 1; une hypersurface normale est régulière). Pour démontrer ce résultat, l'auteur détermine explicitement les formes en question dans un système convenable de coordonnées affines. Il démontre deux lemmes sur les dimensions d'intersection, ainsi que le lemme suivant: soient $F(X_0,\cdots,X_{n+1})$ l'équation de V, m son degré, $i_1,\cdots,i_s$ des indices tels que, en notant F_i la dérivée partielle de F par rapport à X_i, l'ensemble d'équations $F(X)=F_{i_1}(x)=\cdots=F_{i_s}(X)=0$ soit de dimension $n-s$ (s est la codimension de S; les indices en question existent en vertu d'un lemme précédent); alors il n'existe aucune relation non triviale de la forme

$$\sum_{j=1}^{s}A_j(X)F_{i_j}(X)=B(X)F(X),$$

où les $A_j(X)$ sont des formes de degré $<m-1$.
P. Samuel (Clermont-Ferrand)

5204:

Abhyankar, Shreeram. On the ramification of algebraic functions. II. Unaffected equations for characteristic two. Trans. Amer. Math. Soc. **89** (1958), 310–324.

Let V be an irreducible normal algebraic variety of dimension $r\geqq 2$ with function field K/k, where k is an algebraically closed field of characteristic p, and let P be a simple point of V. In an earlier paper [Amer. J. Math. **77** (1955), 575–592; MR **17**, 193] the author has shown that if Q is a point on a normalization of V in a finite algebraic extension L of K which is such that the branch locus D on V has a t-fold normal crossing ($t\leqq r$) at P, then the local Galois group $G(Q/P)$ is a p_t-group. (That is, G/Π is a direct product of at most t cyclic groups, where Π is the normal subgroup generated by all of the p-Sylow groups of G.) In this paper the following existence question is raised. Given a pure $(r-1)$-dimensional subvariety D of V with a t-fold normal crossing at P and a fixed p_t-group G, does there exist an algebraic extension L of K for which the branch locus coincides locally (at P) with D, and a point Q in L over P such that $G(Q/P)=G$? By explicit constructions, the author answers this question in the affirmative for the case $p=2$, $t=1$, $G=S_n$, the symmetric group on n letters. *H. T. Muhly* (Iowa City, Iowa)

5205:

✱Severi, Francesco. Fonctions et variétés quasi-abéliennes. Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 521–528. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

L'A. presenta in sintesi lo sviluppo della teoria delle funzioni quasi abeliane e delle relative varietà quasi abeliane, da lui fondata nel 1947 [F. Severi, Funzioni quasi abeliane, Pontificiae Academiae Scientiarum Scripta Varia, vol. 4, (1947); MR **9**, 578], mettendo in particolare evidenza l'analogia e ad un tempo le profonde differenze esistenti tra le funzioni e le varietà quasi abeliane da una parte e le funzioni e le varietà abeliane dall'altra. Tra le tante ricordiamo soltanto il possesso da parte di una varietà quasi abeliana V_π di un gruppo continuo abeliano di trasformazioni birazionali ∞^n, che è generalmente transitivo, e non assolutamente transitivo come nel caso delle varietà abeliane di Picard. Ne seguono le conseguenze più impreviste, come quella che V_π possiede delle corrispondenze trascendenti, oltrechè algebriche, rap-

presentate da congruenze lineari tra gli integrali virtualmente di prima specie su V_π, circostanza questa che non ha riscontro nel caso abeliano. Nella sua esposizione l'A. ricorda i contributi portati alla teoria da F. Conforto e da M. Benedicty, e mette a fuoco i problemi ancora insoluti indirizzando il lettore ad una loro possibile soluzione.

M. Rosati (Rome)

5206:

✱Severi, Francesco. Problèmes résolus et problèmes nouveaux dans la théorie des systèmes d'équivalence. Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 529–541. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

È uno sguardo d'insieme alla teoria dei sistemi di equivalenza, considerati dall'A. fin dal 1932. Nella prima parte della esposizione l'A. richiama le definizioni e le proprietà fondamentali dell'equivalenza razionale e dei relativi sistemi di equivalenza, mostrando come sia necessaria e naturale la loro introduzione (accanto alla equivalenza algebrica, di cui però egli qui non si occupa) per lo studio delle proprietà birazionali delle varietà algebriche, quando si sia constatata l'insufficienza dell'equivalenza lineare per raggiungere il medesimo scopo. Dopo avere ricordato i contributi dei numerosi autori che hanno lavorato alla costruzione di questa teoria, in particolare i più recenti di B. Segre sulle successioni canoniche di una data varietà, l'A. passa ad illustrare gli intimi legami della teoria dei sistemi di equivalenza con la topologia. Ovunque l'A. ha cura di sottolineare esplicitamente i problemi ancora aperti, indicando anche qualche previsione che possa anticipare i risultati. *M. Rosati* (Rome)

5207:

Mattuck, Arthur. Cycles on abelian varieties. Proc. Amer. Math. Soc. 9 (1958), 88–98.

È uno studio dell'omologia algebrica a coefficienti razionali sopra una varietà abeliana a moduli generali, che estende un precedente risultato di Comessatti [A. Comessatti, Rend. Sem. Mat. Univ. Padova 5 (1934), 50–79]. L'A. dimostra che, per ogni valore della dimensione complessa le classi di omologia rappresentate da sottovarietà algebriche sono soltanto la classe di omologia contenente une sezione lineare di quella dimensione e i suoi multipli razionali. Inoltre le classi di omologia della varietà, che appartengono a queste sottovarietà, sono soltanto quelle ottenute come intersezione delle sottovarietà stesse con le classi di omologia della varietà abeliana. *M. Rosati* (Rome)

LINEAR ALGEBRA

See also 5156, 5160, 5162, 5217, 5221, 5447.

5208:

✱Hohn, Franz E. Elementary matrix algebra. The Macmillan Co., New York, N.Y., 1958. xii+305 pp. $10.00.

This is a text written for upperclassmen and beginning graduate students who wish to have a good foundation in matrix theory for use: 1: in further mathematics; 2: in applications to numerical analysis and the physical sciences; 3: in applications to the social sciences. The author first considers only matrices with real or complex elements and develops the fundamental operations with matrices and determinants. He first defines rank and equivalence in terms of determinants and by means of the elementary transformations. Then, after dealing with linear equations and linear dependence, he considers vector spaces over a number field and their relation to concepts previously dealt with. The closing three chapters are concerned with unitary and orthogonal transformations, the characteristic equation, and bilinear, quadratic, and Hermitian forms. There are three appendices: The notations $\sum$ and $\prod$, The algebra of complex numbers, and The general concept of isomorphism.

The reviewer is disappointed that more indication of applications is not given to relieve the book's somewhat formal character. Not only are there numerous applications which require little background, but many lend insight to the development. For instance, there are many advantages in introducing vector spaces in the beginning and defining rank in their terms; the geometric aspects deserve better treatment.

The book is very clearly written, and there are both formal and thought-provoking exercises. From the rather narrow viewpoint, the pupil is led in gradual stages to the deeper ideas of fields, groups, isomorphism, vector spaces, when he has the background to appreciate them. This should be an excellent introduction to some of the ideas of modern algebra. *B. W. Jones* (Boulder, Colo.)

5209:

Abellanas, Pedro. Matrices of polynomials in several indeterminates. Rev. Mat. Hisp.-Amer. (4) 17 (1957), 267–277. (Spanish)

L'Auteur étudie la réduction canonique diagonale de matrices dont les coefficients sont des polynômes en plusieurs indéterminées, par rapport à des produits de certaines transformations élémentaires. *G. Papy* (Brussels)

5210:

Picone, Mauro. Sulla teoria delle matrici nel corpo complesso. Boll. Un. Mat. Ital. (3) 13 (1958), 1–6.

Le théorème de Bessel est étendu aux matrices et la meilleure borne pour le rang d'une valeur est donnée.

G. Papy (Brussels)

5211:

Cherubino, Salvatore. Sulla teoria delle matrici. Boll. Un. Mat. Ital. (3) 13 (1958), 7–10.

Quelques commentaires sur la notation matricielle et la définition des fonctions holomorphes de matrices.

G. Papy (Brussels)

5212:

Fulton, Curtis M.; and Norton, Donald A. Nonexistence of fixed subspaces under affine transformations. Math. Z. 70 (1958), 52–54.

Let A be a linear transformation in the Euclidean n-space E^n, and $b \in E^n$. Let the transformation T be defined by $Tx=Ax+b$. For each $x \in E^n$, let $V(x)$ denote the determinant whose row vectors are $Tx-x$, T^2x-x, $\cdots$, T^nx-x. It is shown that the following conditions are mutually equivalent: (1) No proper subspace of E^n is invariant under T. (2) $V(x)$ is a non-zero constant independent of x. (3) The minimum equation of $A-I$ is $t^n=0$ and $(A-I)^{n-1}b\neq 0$; where I denotes the identity transformation. *Ky Fan* (Notre Dame, Ind.)

5213:

Mirsky, L. Diagonal elements of orthogonal matrices. Amer. Math. Monthly 66 (1959), 19–22.

A necessary and sufficient condition for n real numbers

$d_1, d_2, \cdots, d_n$ to be the diagonal elements of some orthogonal matrix with determinant $+1$ is $|d_j|\leq 1$ $(1\leq j\leq n)$ and

$$\sum_{j=1}^{n} |d_j|\leq n-2+2s \min_j |d_j|,$$

where s stands for 1 or 0 according as the number of negative terms among $d_1, d_2, \cdots, d_n$ is even or odd. The proof is based on another necessary and sufficient condition previously obtained by A. Horn [Amer. J. Math. **76** (1954), 620–630; MR **16**, 105]. {Reviewer's note: The author writes that "$-d_1, \cdots, -d_n$" on line 4 of page 20 should be replaced by "$-d_1, d_2, \cdots, d_n$".}

Ky Fan (Notre Dame, Ind.)

5214:

Bellman, Richard. Notes on matrix theory. XV. Multiplicative inequalities obtained from additive inequalities. Amer. Math. Monthly **65** (1958), 693–694.

Let A be a positive definite Hermitian matrix of order n, and let $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$ be its eigenvalues. The reviewer [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 652–655; Amer. Math. Monthly **58** (1951), 194; MR **11**, 600] obtained that, for $1\leq k\leq n$,

$$\sum_{i=k}^{n} \lambda_i=\min \sum_{i=k}^{n} (Ax_i, x_i), \quad \prod_{i=k}^{n} \lambda_i=\min \prod_{i=k}^{n} (Ax_i, x_i),$$

where the minimum is taken over all sets of $n-k+1$ orthonormal vectors $x_k, \cdots, x_n$. In the present paper, the author shows that the multiplicative result can be derived from the additive one.

Ky Fan (Notre Dame, Ind.)

5215:

Lineckiĭ, V. D. Solution of a system of trinomial equations by means of focal relations. Leningrad. Inžen.-Stroit. Inst. Naučn. Trudy **17** (1954), 185–190. (Russian)

5216:

Varini, Bruno. Aspetto tensoriale della teoria dei determinanti. Archimede **10** (1958), 71–80.

A continuation of the article in Archimede **9** (1957), 97–104, [MR **19**, 1034].

ASSOCIATIVE RINGS AND ALGEBRAS

See also 5225, 5228.

5217:

Drazin, M. P. Pseudo-inverses in associative rings and semigroups. Amer. Math. Monthly **65** (1958), 506–514.

The author calls an element x in an associative ring R pseudo-invertible if there is some element x' in R, a pseudo-inverse, with (1) $x'x=xx'$, (2) $x^m=x^{m+1}x'$ for some positive integer $m=m(x)$, and (3) $x'=x'^2x$. This concept generalizes the generalized inverse defined for complex matrices by E. H. Moore [General analysis, Amer. Philos. Soc., Philadelphia, Pa., 1935] and Penrose [Proc. Cambridge Philos. Soc. **51** (1955), 406–413; MR **16**, 1082]. He then shows that if for an element x a pseudo-inverse exists, it is unique and commutes with all elements commuting with x. The pseudo-inverses of powers and orthogonal sums are studied, and it is also shown that the condition of pseudo-invertibility is equivalent to the element x being strongly π-regular in the sense of Azumaya [J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1954), 34–39; M.R. **16**, 788], i.e., there exist elements a, b and positive integers p, q with (4) $x^p=ax^{p+1}$ and (5) $x^q=x^{q+1}b$. If only (4) holds, x is called right π-regular. The author then gives a new proof of the following theorem of Azumaya [loc. cit.]: If R has all its nilpotent elements of index $\leq N$, then every right π-regular element is pseudo-invertible with $m(x)\leq N$. Finally, a necessary and sufficient condition for an element of a semi-group to be pseudo-invertible is given.

A. Rosenberg (Evanston, Ill.)

5218:

Feller, Edmund H. Properties of primary noncommutative rings. Trans. Amer. Math. Soc. **89** (1958), 79–91.

In this paper some of Snapper's theory of completely primary commutative rings [Ann. of Math. (2) **52** (1950), 666–693; MR **12**, 314] is extended to "duo" rings: rings in which every left (right) ideal is also a right (left) ideal. After first studying the Jacobson, nil and nilpotent radicals of such rings, the author turns to factorisation questions. The main results are for the case that the ring modulo its nil radical is a principal ideal domain and results very analogous to Snapper's [loc. cit.] are obtained. Snapper's conjecture that the number of irreducible elements in a factorisation is unique is disproved by means of a counter-example. The last section deals with the radicals of $R[x]$, R a duo ring: it is shown, e.g., that the Jacobson radical of $R[x]$ is the set of all nilpotent elements of $R[x]$.

A. Rosenberg (Evanston, Ill.)

5219:

Kurata, Yoshiki. Decompositions of semi-prime rings and Jordan isomorphisms. Osaka Math. J. **9** (1957), 189–193.

A ring is called semi-prime in case the intersection of its family P of prime ideals is zero. If there is a minimal subfamily of P with zero intersection, the author shows that this subfamily is unique and that special subdirect sums of prime rings have this property. Using Herstein's theorem on Jordan homomorphisms onto prime rings [Trans. Amer. Math. Soc. **81** (1956), 331–341; MR **17**, 938], it is shown that if ϕ is a Jordan isomorphism of a ring R onto a semi-prime ring R', then (1) R is semi-prime, (2) each representation of R' as a subdirect sum of prime rings yields a similar representation of R such that corresponding summands are either isomorphic or anti-isomorphic, (3) ϕ maps the center of R onto the center of R'.

M. F. Smiley (Iowa City, Iowa)

5220:

Andrunakievič, V. A. Radicals of associative rings. I. Mat. Sb. N.S. **44(86)** (1958), 179–212. (Russian)

This paper is an elaboration of results which were briefly announced earlier [Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 487–490; MR **19**, 835].

R. A. Good (College Park, Md.)

5221:

Büke, Altintaş. Nilpotente Algebren vom Index 3 über einen Körper K der Charakteristik 2. Rev. Fac. Sci. Univ. Istanbul Sér. A **22** (1957), 45–89; Berichtigung 190. (Turkish summary)

In dieser Arbeit setzt Verf. seine Untersuchungen ueber kommutativ-assoziative nilpotente Algebren vom Index 3 ueber einen Koerper der Charakteristik 2 fort [dieselbe Rev. **19** (1954), Supplement, 1–145; MR **16**, 561]. Zuerst werden Algebren vom Geschlecht (m, n) [ibid.] untersucht und in sog. Typenklassen eingeteilt. Die Algebren von Geschlechtskrankheit $(m, 3)$ werden dann eingehender

betrachtet und alle ihre Typenklassen aufgestellt. Zuletzt gibt Verf. die Multiplikationstafel eines Beispiels dieses Seiten. Geschlechtsgenau an. Dazu benoetigt er nur vierzehn

A. Rosenberg (Evanston, Ill.)

5222:

Širšov, A. I. On the Levitzki problem. Dokl. Akad. Nauk SSSR **120** (1958), 41–42. (Russian)

Let $\Pi(\lambda)=\lambda_{i_1}\cdots\lambda_{i_k}$ be a monomial in some indeterminates $\lambda_1, \cdots, \lambda_t$ and $T_\Pi(\lambda)$ the set of all monomials in these indeterminates of degree $\geqq k$ different from $\Pi(\lambda)$. The (associative) ring S is called strongly pivotal in the sense of M. P. Drazin [Proc. Amer. Math. Soc. **8** (1957), 352–361; MR **18**, 869] if there exists a monomial $\Pi(\lambda)$ such that for every $x_1, \cdots, x_t$ ($\in S$) the element $\Pi(x)$ is contained in the right ideal generated by the set of all elements $\tau(x)$, for $\tau(\lambda)\in T_\Pi(\lambda)$. J. Levitzki [Bull. Amer. Math. Soc. **49** (1943), 462–466; MR **4**, 238] raised the question whether all nilrings are nilpotent. The author answers this question in the affirmative for a rather extensive class of rings. Indeed, he proves that any strongly pivotal (associative) nilring is locally nilpotent.

A. Kertész (Debrecen)

5223:

Leavitt, William G. A ring admitting modules of limited dimension. Proc. Amer. Math. Soc. **9** (1958), 660–664.

A module M over a ring K with a unit is said to be finite dimensional if it is a finitely generated free K-module and it does not contain an infinite set of independent elements. The author has proved previously [An. Acad. Brasil Ci. **27** (1955), 241–250; MR **17**, 578; p. 242] that if M is finite dimensional then all bases of M over K have the same length n; and if for a given ring K, there exists a finite dimensional module with a length of basis n, then every free module over K generated by $\leqq n$ elements is finite dimensional. In the present paper, for each integer n, the author constructs a ring K which admits finite dimensional modules M over K with length of basis $\leqq n$; but there is a free K-module generated by $n+1$ elements which is not finite dimensional, and consequently, all free K-module generated by more than $n+1$ elements are not finite dimensional.

S. A. Amitsur (Notre Dame, Ind.)

5224:

Villamayor, Orlando E. On the semisimplicity of group algebras. Proc. Amer. Math. Soc. **9** (1958), 621–627.

Sufficient conditions are established for a group algebra over a commutative ring to be semisimple. The main result is (1): If G is a free abelian group and K is a commutative ring without nonzero nilpotent elements, then the group ring $K(G)$ is semisimple. This leads to (2): If G is a commutative group and K is a semisimple commutative algebra over the rationals, then $K(G)$ is semisimple, and (3): If G/Z is locally finite, where Z is the center of a group G, and if K is a semisimple commutative algebra over the rationals, then $K(G)$ is semisimple.

Among the lemmas are two: If G/C is locally finite, where C is a subgroup contained in the center of a group G, and K is a commutative ring, then $K(C)$—w. dim $K(G)$ $=0$ if and only if K is uniquely divisible by the order of each element in G/C; and: If C is a subgroup contained in the center of a group G and if K is a commutative ring, then $K(C)$—w. dim $K(G)=0$ implies G/C is a torsion group and K is uniquely divisible by the order of each element in G/C. These extend results of Auslander for the case $C=(1)$ [same Proc. **8** (1957), 658–664; MR **19**, 390].

G. L. Walker (Southbridge, Mass.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

5225:

Lyu, Šao-syuė. On the splitting of infinite algebras. Mat. Sb. N.S. **42(84)** (1957), 327–352. (Russian)

Questions about decomposition, such as Curtis studied [Duke Math. J. **21** (1954), 79–85; MR **15**, 774], are here raised for algebras of more general type. The classes of algebras considered, besides associative algebras over a field of arbitrary characteristic, are alternative algebras, Jordan algebras, and algebraic Lie algebras over a field of characteristic zero. Sufficient conditions are sought for an algebra A to split relative to an ideal R, that is, for a subalgebra P to exist such that $A=P+R$ and $P\cap R=0$; if another subalgebra P' satisfies $A=P'+R$ and $P'\cap R=0$, the relationship between P and P' is analyzed. For a subalgebra of A, let C be the property of being, in the associative case, separable or, in one of the three non-associative cases, semisimple; analogously, let N be the property of being, in the Lie case, solvable or, in one of the other three cases, nilpotent. Let $R_1=R=R^{[1]}$; in the Jordan case, let $R_k=\{R_{k-1}R^2, (R_{k-1}R)R\}$; in the other three cases, let $R_k=R_{k-1}R$ and let $R^{[k]}=\{RR^{[k-1]}, R^{[k-1]}R\}$. The algebra A splits and a locally C-subalgebra of A is weakly conjugate (relative to R) in A to a subalgebra of P if either of the following two sets of conditions holds: (i) R is a locally N-radical, and A/R is a locally C-algebra of at most countable rank; (ii) $\bigcap_{n=1}^{\infty} R_n=0$, A is complete in the topology defined by $\{R_n\}$, and A/R is a locally C-algebra of at most countable rank. The algebra A splits and P' is conjugate in A to P if either of the following two sets of conditions holds: (i) A is a Jordan algebra, R is the locally N-radical of A, R is finite, and A/R is a direct sum of arbitrarily many finite C-algebras; (ii) A is an associative or alternative or algebraic Lie algebra, $\bigcap_{n=1}^{\infty} R^{[n]}=0$, R is a linearly compact A-module in the topology defined by $\{R^{[n]}\}$, and A/R is a direct sum of arbitrarily many finite C-algebras.

R. A. Good (College Park, Md.)

5226:

Mutō, Yosio. On linear Lie algebras. I. J. Math. Soc. Japan **10** (1958), 160–183.

This is to be the first of a series of papers studying some properties of subalgebras of the linear Lie algebra $\mathfrak{gl}(R^n)$ formed by the set of all linear transformations X on an n-dimensional linear space R^n over the field of real numbers with law of composition $[X, Y]=XY-YX$. Most of the paper is devoted to obtaining bases with special properties for the subalgebras. If $\mathfrak{g}$ is an r-dimensional subalgebra there is an isomorphic representation $\mathfrak{K}(\mathfrak{g}, S)$ in the linear space of all n by n matrices, where S is a basis of R^n. If $\mathfrak{V}(\mathfrak{g}, S)$ is the $m=n^2-r$ dimensional subspace of matrices $(V^{\lambda}_{\cdot\mu})$ such that for all matrices $(K^{\lambda}_{\cdot\mu})$ in $\mathfrak{K}(\mathfrak{g}, S)$, $K^{\lambda}_{\cdot\mu}V^{\mu}_{\cdot\lambda}=0$ (the summation convention has been adopted), then S can be selected so that $\mathfrak{V}$ can be decomposed and $\mathfrak{V}=\mathfrak{V}_1+\cdots+\mathfrak{V}_T+\mathfrak{V}_{\hat{T}}$ $(1\leqq T\leqq P)$. To get from the decomposition for T to that for $T+1$, $\mathfrak{V}_{\hat{T}}$ is decomposed to obtain $\mathfrak{V}_{\hat{T}}=\mathfrak{V}_T+\mathfrak{V}_{\widehat{T+1}}$; P denotes the last possible step in the decomposition. The bases of the summands are described in detail. There are integers n_T associated with $\mathfrak{V}_T$ and putting $n_0=n$, $d_T=n_{T-1}-n_T$, $d_{P+1}=n_P$, then $n=d_1+\cdots+d_{p+1}$ and $\mathfrak{V}_T$ has dimension $d_T(n-d_1-\cdots-d_T)$ or $d_T(n-d_1-\cdots-d_T)+1$. The ordered set $d_1, \cdots, d_{p+1}$ is called the d series of $\mathfrak{g}$. Inequalities involving the d series are obtained and will

enable the author to find all subalgebras $\mathfrak{g}$ of sufficiently large dimension r and to get some properties of $\mathfrak{g}$ of more general dimensions. *L. A. Kokoris* (Chicago, Ill.)

5227:

Boers, A. H. L'anneau 5-alternatif. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 532–534.

For the terminology employed in this paper, see the author's thesis ["Généralisation de l'associateur", J. Van Tuyl, Antwerpen-Zaltbommel, 1957; MR **19**, 9]. The following theorem is proved: A 5-alternative ring of characteristic $\neq 2$ is associative in case $x^2=0$ implies $x=0$.
R. D. Schafer (Princeton, N.J.)

HOMOLOGICAL ALGEBRA

See also 5194, 5485.

5228:

MacLane, Saunders. Extensions and obstructions for rings. Illinois J. Math. 2 (1958), 316–345.

Une extension d'algèbres est définie par un épimorphisme d'algèbres $E\to\Lambda$; lorsque le produit de 2 éléments quelconques du noyau A est 0, la multiplication à gauche (resp. à droite) par un élément de E induit sur A une structure de Λ-bimodule. Ce cas a été étudie par Hochschild lorsqu'il s'agit d'algèbres sur un corps: étant donné l'algèbre Λ et un Λ-bimodule A, l'ensemble des classes d'extensions qui définissent sur A cette structure de bimodule est en correspondance bijective avec le groupe de cohomologie de Hochschild $H^2(\Lambda, A)$. Ici, comme dans un travail antérieur [Colloq. topologie algébrique, Louvain, 1956, pp. 55–80, Thone, Liège; Masson, Paris; 1957; MR **20** #892], l'auteur cherche d'une part à s'affranchir de l'hypothèse suivant laquelle les algèbres considérées ont un corps de base (en fait, il étudie ici le cas des anneaux, c'est-à-dire des algèbres sur Z), ce qui introduit des obstructions de caractère additif à côté des obstructions multiplicatives; d'autre part, il étudie aussi le cas où la multiplication induite sur A n'est pas nulle. Dans ce but, il donne une définition nouvelle des groupes de cohomologie $H^n(\Lambda, K)$ à coefficients dans un Λ-bimodule K (cf. § 5). On prouve (th. 3): si K est un Λ-bimodule, les classes d'extensions $0\to K\to E\to\Lambda\to 0$ qui définissent sur K cette structure de bimodule (la multiplication induite sur K étant nulle) sont en correspondance bijective avec $H^2(\Lambda, K)$.

Dans le cas général, soit A une algèbre (sans élément unité); on appelle "bimultiplication" un couple (σ, σ') formé d'un endomorphisme σ de A_d et d'un endomorphisme σ' de A_g (où A_d, resp. A_g, désigne A muni de sa structure de A-module à droite, resp. à gauche) tels que $a(\sigma b)=(a\sigma')b$ pour $a, b\in A$. Les bimultiplications forment un anneau M_A, en définissant $(\sigma, \sigma')\cdot(\tau, \tau')=(\sigma\tau, \sigma'\tau')$; on a un homomorphisme $\mu: A\to M_A$ qui associe à $a\in A$ le couple $(x\to ax, x\to xa)$, d'où une suite exacte

$$0\to K_A\to A\overset{\mu}{\to} M_A\to P_A\to 0,$$

où P_A est un anneau, car l'image de μ est un idéal bilatère. K_A s'appelle le bicentre de A. La donnée d'une extension d'algèbres $0\to A\to E\to\Lambda\to 0$ définit un homomorphisme $\theta:\Lambda\to P_A$ dont l'image se compose d'éléments 2 à 2 permutables (on dit que (σ, σ') et (τ, τ') sont permutables si $\sigma(a\tau')=(\sigma a)\tau'$, $\tau(a\sigma')=(\tau a)\sigma'$; cette notion passe au quotient dans P_A); on en déduit une structure de Λ-bimodule sur le bicentre K_A. On a les théorèmes suivants: tout homomorphisme $\theta:\Lambda\to P_A$ dont l'image se compose d'éléments 2 à 2 permutables définit une obstruction $\xi_\theta\in H^3(\Lambda, K_A)$; pour que θ provienne d'une extension, il faut et il suffit que $\xi_\theta=0$, et alors les classes d'extensions qui réalisent θ sont en correspondance bijective avec $H^2(\Lambda, K_A)$; on peut définir directement la structure de groupe abélien de l'ensemble de ces classes d'extensions. Étant donnés un Λ-bimodule K et un élément $\xi\in H^3(\Lambda, K)$, cherchons s'il existe un A ayant pour bicentre K, et un $\theta:\Lambda\to P_A$ qui induise sur K la structure donnée de Λ-bimodule et soit tel que $\xi_\theta=\xi$; une condition nécessaire et suffisante (th. 8) est que ξ soit dans le noyau d'un certain homomorphisme

$$\bar{\gamma}: H^3(\Lambda, K)\to \mathrm{Hom}(\Lambda/2\Lambda, K).$$

Ce dernier résultat suggère que la définition des groupes de cohomologie $H^n(\Lambda, K)$ choisie par l'auteur n'est peut-être pas la bonne pour toutes les dimensions. Ils sont définis comme suit: un certain "complexe cubique" Q, déjà étudié par Eilenberg-MacLane, est appliqué à Λ et donne une algèbre différentielle graduée $Q(\Lambda)$ avec une augmentation $\eta: Q(\Lambda)\to\Lambda$, telle que

$$H_0(Q(\Lambda))\approx\Lambda,\ H_1(Q(\Lambda))=0,\ H_2(Q(\Lambda))=\Lambda/2\Lambda;$$

$Q(\Lambda)$ est donc "acyclique" en dimensions 0 et 1, mais pas au-delà. Ensuite, on applique à $Q(\Lambda)$, munie de l'augmentation η, la "bar construction" convenablement adaptée, ce qui donne un Λ-module différentiel $\bar{B}(Q(\Lambda), \eta)$; on pose alors

$$H^n(\Lambda, K)=H^n(\mathrm{Hom}_\Lambda(\bar{B}(Q(\Lambda), \eta), K).$$

Dans tout ce machinisme, on fait jouer au foncteur $Q(\Lambda)$ un rôle qui n'est probablement pas justifié. L'auteur indique lui-même comment $Q(\Lambda)$ pourrait être remplacé, en basses dimensions, par une autre résolution $U(\Lambda)$. Le rapporteur est convaincu que des recherches fructueuses doivent être poussées dans cette direction.

Erratum: p. 325, ligne 2 du bas, au lieu de $\theta: A\to M_K$, lire $\theta:\Lambda\to M_K$. Page 326, ligne 1 du § 7, au lieu de "For given rings", lire: "For given maps".
H. Cartan (Paris)

5229:

Eilenberg, Samuel; Rosenberg, Alex; and Zelinsky, Daniel. On the dimension of modules and algebras. VIII. Dimension of tensor products. Nagoya Math. J. **12** (1957), 71–93.

The authors carry out a systematic study of the dimension of the tensor product $\Lambda\otimes_K\Gamma$ of K-algebras Λ and Γ. The major weapon used is a battery of three fundamental spectral sequences. Complete results are obtained when Γ is a ring of matrices, triangular matrices, polynomials or rational functions.

The spectral sequences used are the following:

(I) $H^p(\Gamma, \mathrm{Ext}_\Lambda{}^q(B, C))\underset{p}{\Rightarrow}\mathrm{Ext}_{\Lambda\otimes\Gamma}{}^n(B, C)$ (valid when Γ is K-flat);

(II) $H^p(\Gamma, H^q(\Lambda, A))\underset{p}{\Leftarrow}H^n(\Lambda\otimes\Gamma, A)$ (valid when Γ is K-flat);

(III) $H_K{}^p(\Lambda, H_L{}^q(K, A))\underset{p}{\Rightarrow}H_L{}^n(\Lambda, A)$

(where K is an L-algebra and $\mathrm{Tor}_r{}^{K\otimes_L K}(\Lambda\otimes_L\Lambda^*, K)=0$ for $r>0$). The subcripts K, L are used to denote which ring is the ground ring over which the cohomology is being taken.

Some of the main theorems obtained using these spectral sequences are the following.

(a) If $\Gamma=K[x_1, \cdots, x_n]$ (x_i indeterminates), then dim $\Gamma=n$. Moreover, if Λ is any K-algebra, we have: i) l.gl. dim $\Lambda\otimes\Gamma=n+$l.gl. dim Λ; ii) dim $\Lambda\otimes\Gamma=n+$dim Λ; iii) if K is an L-algebra, then L-dim $\Gamma=n+L$-dim K; and iv) if Λ is a Γ-algebra such that $H_r^K(\Gamma, \Lambda\otimes_K\Lambda^*)=0$ for $r>0$ and Γ-dim $\Lambda<\infty$, then K-dim $\Lambda=n+\Gamma$-dim Λ.

(b) If $\Gamma=K(x_1, \cdots, x_n)$, then dim $\Gamma=n$ and ii), iii), and iv) are also true.

(c) If $\Gamma=T_n(K)$, $n>1$, is the algebra of all $n\times n$ triangular matrices over K, then dim $\Gamma=1$ and i), ii), iii) above are true with n replaced by 1.

Other results are obtained by applying spectral sequences and also by applying products. Some theorems are also proved about semi-primary rings, although spectral sequences do not seem to play an important role here. *D. Buchsbaum* (Providence, R.I.)

GROUPS AND GENERALIZATIONS

See also 5217, 5446, 5447.

5230:

Boone, William W. Certain simple, unsolvable problems of group theory. IV. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 571–577.

5231:

Boone, William W. Certain simple, unsolvable problems of group theory. V, VI. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 22–27, 227–232.

A proof is given of the unsolvability of the Word Problem in the theory of groups. A group is defined, in terms of a finite number of generators and a finite number of defining relations, such that the set of all those formal products of generators and their inverses that represent the identity element is not recursive. The proof is independent of, and different from, an earlier proof by P. S. Novikov [Trudy Mat. Inst. Steklov. no. **44** (1955); MR **17**, 706]. The method is a refinement of that used by the author in the earlier papers of this series [same Proc. **57** (1954), 231–237, 492–497; **58** (1955), 252–256; MR **16**, 564, and Part IV, listed above] to obtain a weaker result. The present proof does not depend on the earlier result, and is in some respects a simplification of the previous argument; however, the exposition borrows heavily from the previous parts.

The starting point is a result of Turing [Proc. London Math. Soc. (2) **42** (1936), 230–265], that there exists a 'machine' for which it is undecidable, given arbitrary initial conditions, whether it will ever enter a given internal configuration. Following Post [J. Symb. Logic **12** (1947), 1–11; MR **8**, 558], this result leads to a certain formal system, consisting of words, or strings of symbols, and rules of transformation, for which it is not decidable whether a given word is transformable into one in a certain special class. This formal system is successively modified, first by introducing formal inverses to the original symbols and by making the transformation rules reversible, and finally by a crucial modification corresponding to converting the original Turing machine into a two-phase machine. The undecidability is shown to be preserved, by extensions of methods of Malcev [Mat. Sb. N.S. **6(48)** (1939), 331–336; **8(50)** (1940), 251–264; MR **2**, 7, 128] and Turing [Ann. of Math. (2) **52** (1950), 491–505; MR **12**, 239]. Ultimately a system is obtained for which the set of all words in certain symbols, when identified under the relation of intertransformability, constitute under juxtaposition a group with undecidable word problem. *R. C. Lyndon* (Ann Arbor, Mich.)

5232:

McKay, James H. Another proof of Cauchy's group theorem. Amer. Math. Monthly **66** (1959), 119.

Cauchy's theorem that a group G of order n divisible by p has kp solutions to the equation $x^p=1$ is shown to follow from a simple counting process applied to the n^{p-1} elements of the set S made up of p-tuples $a_1, a_2, \cdots, a_p$, with the property that $a_1a_2\cdots a_p=1$.

R. G. Stanton (Waterloo, Ont.)

5233:

Suh, Tae-Il. Note on a lattice-isomorphism between finite groups with complete partitions. Kyungpook Math. J. **1** (1958), 33–36.

A partition P of a group G is a decomposition of G into the set union of subgroups H_i which overlap only in the identity; P is called complete if each H_i is cyclic, and then G is called completely decomposed (c.d.) [cf., e.g., P. G. Kontorovič, Mat. Sb. **5(47)** (1939), 289–296; MR **2**, 3; M. Suzuki, J. Math. Soc. Japan **2** (1950), 165–185; MR **13**, 907]. Suppose G is finite and G' is a lattice-isomorphic image of G (i.e., the lattices of subgroups of G and G' are isomorphic).

Theorems: If G is c.d., then G' is c.d.; if G is also solvable, then so is G'. If G is nilpotent and c.d., but not an abelian group of type $(p, \cdots, p)$, then G' is nilpotent. If G is an abelian group of type $(p, \cdots, p)$, then G' is nilpotent if and only if any two elements of distinct orders commute. If G has order greater than p and is not an abelian group of type $(p, \cdots, p)$, but each element has order p, then G' is group-isomorphic to G.

An example is given of a non-solvable non-simple c.d. group. {For surveys of knowledge about lattice-isomorphisms of groups, see M. Suzuki, "Structure of a group and the structure of its lattice of subgroups", Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956 [MR **18**, 715] and G. Zappa, Questioni relativo al reticolo dei sottigruppi di un gruppo: conoscenze attuali e problemi aperti. Convegno italo-francese di algebra astratti, Padova, aprile, 1956, pp. 49–58, Edizioni Cremonese, Rome, 1957 [MR **20** #915].} *P. M. Whitman* (Baltimore, Md.)

5234:

Čunihin, S. A. Sets of non-special subgroups and p-nilpotency of finite groups. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 654–656. (Russian)

A finite group whose order is divisible by a prime p is called a pd-group. A finite non-special group [or pd-group] all of whose nontrivial subgroups [or pd-subgroups] are special is a group of type S [or, respectively, type S_p] [Mat. Sb. **40** (1933), 39–41; ibid. N.S. **4(46)** (1938), 521–528]. Let a nonempty set Π of (some of the) prime divisors of the order g of the group $\mathfrak{G}$ have k members; any collection Λ of subgroups of $\mathfrak{G}$ which are (if there are more than one) pairwise not isomorphic is called a Π-set provided there is a mapping of Π on Λ such that, for each $p\in\Pi$, the image of p is a pd-subgroup. A Π-set of subgroups all of whose members have type S [or type S_p] is called a ΠS-set [or, respectively, a ΠS_p-set]. A ΠS-set all of whose members are p-special is called oriented. Every non-special pd-group $\mathfrak{G}$ has a subgroup $\mathfrak{H}$ of type S_p as

follows: if $\mathfrak{G}$ is not p-nilpotent, then $\mathfrak{H}$ is a p-special pd-group of type S; if $\mathfrak{G}$ is p-nilpotent but not p-special, then $\mathfrak{H}$ is a p-nilpotent pd-group of type S; if $\mathfrak{G}$ is both p-nilpotent and p-special, then $\mathfrak{H}$ is a direct product of a cyclic group of order p and a group of type S whose order is not divisible by p. If $\mathfrak{G}$ is not nilpotent relative to every $p \in \Pi$, then $\mathfrak{G}$ has at least one oriented ΠS-set containing k subgroups. A non-special group has at least one ΠS_p-set containing no less than $k-1$ subgroups. A group which does not have a ΠS_p-set of nontrivial subgroups containing k subgroups is a Π-solvable group. Let $\mathfrak{H}$ and $\mathfrak{H}_1$ be normal subgroups of $\mathfrak{G}$, let $\mathfrak{H} \subseteq \mathfrak{H}_1 \cap \Phi(\mathfrak{G})$, where $\Phi(\mathfrak{G})$ is the Frattini subgroup of $\mathfrak{G}$; let $\mathfrak{H}_1$ contain a subgroup $\mathfrak{N}$ of order n such that $\mathfrak{H}_1 = \mathfrak{N}\mathfrak{H}$ and such that every subgroup of order n which is conjugate in $\mathfrak{G}$ to $\mathfrak{N}$ is conjugate to $\mathfrak{N}$ in $\mathfrak{H}_1$; then $\mathfrak{N}$ is invariant in $\mathfrak{G}$.

R. A. Good (College Park, Md.)

5235a:

Yacoub, K. R. On semi-special permutations on $[2p^\alpha]$. Publ. Math. Debrecen **5** (1958), 246–255.

5235b:

Yacoub, K. R. Semi-special permutations on $[pqr]$. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 217–222.

5235c:

Yacoub, K. R. A note on semi-special permutations. Proc. Glasgow Math. Assoc. 3 (1958), 164–169.

On the set of the group $[n]$ of integers, modulo n, the author considers the semi-special permutations, defined as those permutations ϕ which have the properties (i) $\phi(0)=0$ and (ii) for each $u \in [n]$, ϕ_u, given by $\phi_u(x)=\phi(u+x)-\phi(u)$ for $x \in [n]$, is a power of ϕ with exponent depending on u. Although the automorphisms of $[n]$ are semi-special permutations, the author is more concerned with the remaining ones, the non-linear semi-special permutations [Proc. Glasgow Math. Assoc. 2 (1955), 116–123; 3 (1956), 18–35; Duke Math. J. **24** (1957), 455–465; MR **17**, 11, **19**, 5, 1037]. In the first paper herein reviewed, the author describes all the non-linear ones on $[2p^\alpha]$ by highly technical congruences and then specializes to the simpler results for $\alpha \leq 3$. In the second paper, the results on $[pqr]$, where p, q and r are distinct odd primes, are somewhat less formidable. The author has shown [Proc. Glasgow Math. Assoc. 3, loc. cit.] that each non-linear ϕ equals some ϕ_s and maps onto an automorphism of $[s]$. In the third paper, the author is concerned with the kernel of this map and with its complement. The non-linear semispecial permutations are suitably provided with a set of three or four parameters (one of which is always s). In particular, necessary and sufficient conditions are found for ϕ to lie in the kernel for one s and in the complement of the kernel for another value of s.

F. Haimo (St. Louis, Mo.)

5236:

Žmud', È. M. On kernels of homomorphisms of linear representations of finite groups. Mat. Sb. N.S. **44(86)** (1958), 353–408. (Russian)

This paper is an elaboration and a generalization of a previous paper on isomorphic representations of finite groups [Mat. Sb. N.S. **38(80)** (1956), 417–430; MR **18**, 13; see also the corrections below]. Let $\mathfrak{G}$ be a finite group and k a given positive integer. A "k-kernel" of $\mathfrak{G}$ is defined to be the kernel of a representation of $\mathfrak{G}$ which is composed of exactly k irreducible components. (The ground field P of the representations is an arbitrary field whose characteristic is either 0 or a prime not dividing the group order; however, the following results are independent of the choice of P.) One of the main results states that the number of k-kernels of $\mathfrak{G}$ is equal to the number of normal subgroups of $\mathfrak{G}$ which can be generated by k classes of conjugate elements of $\mathfrak{G}$. This result and others follow from the study of a certain commutative semisimple algebra $\mathfrak{C}_k$ over P which, in the present theory, plays a rôle similar to that of the center of the group algebra in ordinary representation theory.

Let $\mathfrak{G}^{(k)}=\mathfrak{G}\times\cdots\times\mathfrak{G}$ be the k-fold product of $\mathfrak{G}$. Two elements $(G_1, \cdots, G_k)$ and $(H_1, \cdots, H_k)$ of this product are called weakly equivalent if the two normal subgroups of $\mathfrak{G}$ generated by the G_i's and the H_i's, respectively, coincide. If the classes of weakly equivalent elements of $\mathfrak{G}^{(k)}$ are identified with the corresponding sums in the group algebra $\mathfrak{R}^{(k)}$ of $\mathfrak{G}^{(k)}$ over P, then they can be shown to form a basis of a semisimple subalgebra of the center of $\mathfrak{R}^{(k)}$: this is the algebra $\mathfrak{C}_k$. The main result stated above now follows from the fact that $\mathfrak{C}_k$ splits completely over P, and that the minimal idempotents of $\mathfrak{C}_k$ correspond one-one to the k-kernels of $\mathfrak{G}$. To exhibit this correspondence, the author proves the following theorem. For any normal subgroup $\mathfrak{H}$ of $\mathfrak{G}$, denote by $j_{\mathfrak{H}}$ the corresponding idempotent in the group algebra $\mathfrak{R}$ of $\mathfrak{G}$ over P; let $j_{\mathfrak{H}}^{(k)}=j_{\mathfrak{H}}\otimes\cdots\otimes j_{\mathfrak{H}}$ be its k-fold product in $\mathfrak{R}^{(k)}$ (note that $\mathfrak{R}^{(k)}$ is the k-fold tensor product of $\mathfrak{R}$ over P). Put

$$w_{\mathfrak{H}}^{(k)}=\prod_{\mathfrak{F}} (j_{\mathfrak{H}}^{(k)}-j_{\mathfrak{F}}^{(k)}),$$

where $\mathfrak{F}$ ranges over the minimal normal subgroups of $\mathfrak{G}$ containing $\mathfrak{H}$. If $\mathfrak{H}$ is a k-kernel of $\mathfrak{G}$, it is proved that $w_{\mathfrak{H}}^{(k)}$ is a minimal idempotent of $\mathfrak{C}_k$, and that $\mathfrak{H}\Leftrightarrow w_{\mathfrak{H}}^{(k)}$ gives the one-one correspondence mentioned above. Furthermore, it is proved that if $\mathfrak{H}$ is not a k- kernel then $w_{\mathfrak{H}}^{(k)}=0$; hence $w_{\mathfrak{H}}^{(k)}\neq 0$ is a necessary and sufficient condition for a normal subgroup $\mathfrak{H}$ of $\mathfrak{G}$ to be a k-kernel. This condition can also be stated in a numerical form: namely, the coefficient with which the unit element of $\mathfrak{G}^{(k)}$ occurs in $w_{\mathfrak{H}}^{(k)}$ should be $\neq 0$. This coefficient is computed to be equal to

$$\sum_T \mu(T)(\mathfrak{G}:\mathfrak{C}_T)^k,$$

where T ranges over the sets of minimal normal subgroups of $\mathfrak{G}$ containing $\mathfrak{H}$ (including the empty set $\emptyset$); $\mathfrak{C}_T$ denotes the group generated by the groups of the set T (with $\mathfrak{C}_\emptyset=\mathfrak{H}$); $\mu(T)=(-1)^t$, where t is the number of elements in the set T. — There is a certain dual result, obtained by means of the characters of the algebra $\mathfrak{C}_k$, and giving a necessary and sufficient condition for a normal subgroup $\mathfrak{H}$ of $\mathfrak{G}$ to be generated by k classes of conjugate elements in $\mathfrak{G}$. This condition requires that $\sum_S \mu(S)(\mathfrak{D}_S:1)^k\neq 0$, where S ranges over the sets of maximal normal subgroups of $\mathfrak{G}$ contained in $\mathfrak{H}$, and $\mathfrak{D}_S$ denotes the intersection of the groups of the set S. Both these conditions lead to the author's generalization of Gaschütz's theorem (see the paper mentioned above).

Now let $k=1$; then the algebra $\mathfrak{C}=\mathfrak{C}_1$ is a subalgebra of the center of $\mathfrak{R}$. It is shown that the structure of $\mathfrak{C}$ determines the structure of the lattice N of normal subgroups of $\mathfrak{G}$, and conversely. More precisely: Let $\bar{\mathfrak{G}}$ be another finite group with corresponding $\bar{\mathfrak{C}}$, $\bar{N}$. Consider those algebra isomorphisms $\mathfrak{C}\to\bar{\mathfrak{C}}$ which map the basis elements of $\mathfrak{C}$ (i.e., the classes of weakly equivalent elements in $\mathfrak{G}$) onto the basis elements of $\bar{\mathfrak{C}}$ ("basis isomorphisms"). On the other hand, consider those lattice isomorphisms $N\to\bar{N}$ which preserve the group order

("order isomorphisms"). Then the basis isomorphisms $\mathfrak{C}\to\overline{\mathfrak{C}}$ correspond one-one to the order isomorphisms $N\to\overline{N}$. The proof of this theorem is obtained by showing that $\mathfrak{C}$ is generated by the idempotents $j_{\mathfrak{H}}$ with $\mathfrak{H}$ in N. — There is a certain dual theorem concerning the lattice anti-isomorphisms $N\to\overline{N}$ which carry the group order into the group index (the latter is to be understood in the full group $\overline{\mathfrak{G}}$). These anti-isomorphisms correspond one-one to the basis isomorphisms of $\mathfrak{C}$ onto a certain subalgebra $\overline{\mathfrak{C}}^*$ of the character algebra of $\overline{\mathfrak{G}}$. This algebra may be described (for the group $\mathfrak{G}$ instead of $\overline{\mathfrak{G}}$) as follows: For each 1-kernel $\mathfrak{H}$ of $\mathfrak{G}$, consider the irreducible characters χ of $\mathfrak{G}$ with $\mathfrak{H}$ as kernel, let ν_χ be the multiplicity of χ in the regular representation of $\mathfrak{G}$; then put $s_{\mathfrak{H}}=\sum_\chi \nu_\chi\chi$. These functions $s_{\mathfrak{H}}$ can be shown to form the basis of a semi-simple subalgebra of the character algebra of $\mathfrak{G}$: this is the algebra $\mathfrak{C}^*$. It is proved that $\mathfrak{C}^*$ splits completely over P, and that the minimal idempotents of $\mathfrak{C}^*$ correspond one-one to the normal subgroups of $\mathfrak{G}$ which can be generated by one class of conjugate elements in $\mathfrak{G}$.

For arbitrary k, there are similar functions $s_{\mathfrak{H}}^{(k)}$, defined on $\mathfrak{G}^{(k)}$ by

$$s_{\mathfrak{H}}^{(k)}=\sum \nu_1\chi_1\otimes\cdots\otimes\nu_k\chi_k,$$

where the sum is to be extended over all those k-tuples $\chi_1, \cdots, \chi_k$ of irreducible characters of $\mathfrak{G}$, for which $\chi_1+\cdots+\chi_k$ has kernel $\mathfrak{H}$. These functions $s_{\mathfrak{H}}^{(k)}$ are constant on the classes of weakly equivalent elements, and they induce in the usual way the characters of the algebra $\mathfrak{C}_k$. It is proved that $s_{\mathfrak{H}}^{(k)}(G_1, \cdots, G_k)\neq 0$ if and only if all the G_i belong to the sockel of $\mathfrak{G}$ modulo $\mathfrak{H}$, i.e., the group generated by the minimal normal subgroups of $\mathfrak{G}$ containing $\mathfrak{H}$. The author obtains a product decomposition of $s_{\mathfrak{H}}^{(k)}$ which corresponds to the direct decomposition of the sockel into characteristic subgroups. In particular, this gives a product decomposition of the degree $n_{\mathfrak{H}}^{(k)}$ of $s_{\mathfrak{H}}^{(k)}$; it coincides with the author's product formula given in his earlier paper for the case $\mathfrak{H}=1$. In general, $s_{\mathfrak{H}}^{(k)}$ is a rational integer which is divisible by $n_{\mathfrak{H}}^{(k)}$. *P. Roquette* (Hamburg)

5237a:

Wan, Cheh-hsian [Wan, Zhe-xian]. On the automorphism of linear groups over a non-commutative principal ideal domain of characteristic $\neq 2$. Acta Math. Sinica 7 (1957), 533–573. (Chinese. English summary)

5237b:

Wan, Zhe-xian. On the automorphisms of linear groups over a non-commutative principal ideal domain of characteristic $\neq 2$. Sci. Sinica 7 (1958), 885–933.

[Chinese and English versions of the same paper.] Detailed proofs of theorems announced earlier [Sci. Record (N.S.) 1 (1957), no. 1, 5–8; MR 20#909]. Let R_1, R be noncommutative rings of characteristic 2, $n\geq 3$; R Euclidean; R_1 principal ideal. (1) All automorphisms of the group $\mathrm{SL}_n[R]$: $\{I+re_{ij}, r\in R\}$ are given by $X\to AX^\sigma A^{-1}$, $X\to A(X^\tau)'^{-1}A^{-1}$, where $A\in\mathrm{GL}_n[R]$, σ is an automorphism, τ an anti-automorphism of R. (2) Every isomorphism of $\mathrm{SL}_n[R_1]$ into an isomorphic group in $\mathrm{GL}_n[R_1]$ is of one of the above types. (3) Let χ map $\mathrm{GL}_n[R]$ homomorphically into the center of the multiplicative semigroup of R, and suppose that whenever ν is in this center, $\chi(\nu I)\neq\nu^{-1}$ (except for $\nu=1$). Then every automorphism of $\mathrm{GL}_n[R]$ is given by the composition of of some $\chi(X)$ with the above mappings: $X\to\chi(X)AX^\sigma A^{-1}$, etc. [See also Landin and Reiner, Ann. of Math. (2) 65 (1957), 519–526; MR 19, 388.]

J. L. Brenner (Menlo Park, Calif.)

5238:

Coxeter, H. S. M. On subgroups of the modular group. J. Math. Pures Appl. (9) 37 (1958), 317–319.

The theorem due to K. Goldberg [J. Washington Acad. Sci. 46 (1956), 337–338; MR 19, 123] that every abelian subgroup of the modular group is cyclic is reproved here by geometric arguments, namely by representing the displacements in the hyperbolic plane by elements of the modular group. {Another proof of the same theorem using abstract group theory appeared in the meantime [see A. Karrass and D. Solitar, Proc. Amer. Math. Soc. 9 (1958), 217–221; MR 20#2373]. For generalizations see O. Taussky and J. Todd [J. Washington Acad. Sci. 46 (1956), 373–375; MR 19, 123] and E. C. Dade [Illinois J. Math. 3 (1959), 11–27; MR 20#6463].}

O. Taussky-Todd (Pasadena, Calif.)

5239:

Schiek, Helmut. Über eine spezielle Reihe von Normalteilern. Arch. Math. 9 (1958), 236–240.

Let G be an arbitrary group, $\{x\}$ the free group generated by x, and $G[x]$ the free product of G and $\{x\}$. Put $C_0=G[x]$ and let C_i be defined for any natural i as the normal subgroup generated by G in C_{i-1}. Then $\bigcap_i C_i=G$. For the commutator groups $K_i=[G, C_{i-1}]$ the relation $\bigcap_i K_i=[G, G]$ holds. *A. Kertész* (Debrecen)

5240:

Černikov, S. N. On layer-finite groups. Mat. Sb. N.S. 45(87) (1958), 415–416. (Russian)

Every thin layer-finite group is a subgroup of a thin layer-finite direct product of finite groups. [See same Sb. 22(64) (1948), 101–133; MR 9, 566].

R. A. Good (College Park, Md.)

5241:

Berman, S. D. Groups of which all representations are monomial. Dopovidi Akad. Nauk Ukrain. RSR 1957, 539–542. (Ukrainian. Russian and English summaries)

Let G be a group of finite order, let $\overline{K}$ be an algebraically closed field, the characteristic of which does not divide the order of G. Let K be the arbitrary subfield of $\overline{K}$ and let $R(G, K)$ denote the group algebra of the group G over the field K.

The author determines the conditions necessary and sufficient to make all representations of G over the field K monomial. On the assumption that G and H are groups all the representations of which are monomial, the author defines the conditions of isomorphism of algebras $R(G, \overline{K})$ and $R(H, \overline{K})$, the conditions of isomorphism of the centres of $R(G, K)$ and $R(H, K)$ and the conditions of isomorphism of the algebras $R(G, K)$ and $R(H, K)$, where K is a field of characteristic $p>0$. *Author's summary*

5242:

Makar, Ragy H. On the analysis of the representations of the linear group of dimension 2. Nederl. Akad. Wetensch. Proc. Ser. A 61=Indag. Math. 20 (1958), 475–479.

This paper is an extension of one by the author and S. A. Missiha [same Proc. 61 (1958), 77–93; MR 20#3219] and gives the reduction of $\{m\}\otimes\{\nu\}$, where (ν) is any partition of 5. *G. de B. Robinson* (Toronto, Ont.)

5243:

Farahat, H. K. On Schur functions. Proc. London Math. Soc. (3) 8 (1958), 621–630.

The duality between S_n and GL(d) establishes a correspondence between the ordinary characters of S_n and those of GL(d), which are the Schur functions $\{\lambda\}$. After proving that $\{\lambda\}$ is an irreductible polynomial in the homogeneous symmetric functions, the author translates certain character relations of S_n which are basic in the modular representation theory [cf. Robinson, Amer. J. Math. **70** (1948), 277–294; MR **10**, 678; Thrall and Robinson, ibid. **73** (1951), 721–724; MR **13**, 205; Staal, Canadian J. Math. **2** (1950), 79–92; MR **11**, 415] into relations between Schur functions, giving independent proofs. *G. de B. Robinson* (Toronto, Ont.)

5244:

Morris, A. O. Spin representation of a direct sum and a direct product. J. London Math. Soc. 33 (1958), 326–333.

The basic spin character $\zeta(A)$ of an orthogonal matrix A is given by

$$\zeta(A)=\prod (2 \cos \tfrac{1}{2}\phi_r) \quad (r=1, \cdots, v),$$

where the variable roots of A are denoted by $\exp(\pm i\phi_r)$ ($r=1, \cdots, v$). The author derives formulae which express the spin characters of the direct sum and the direct product of two orthogonal matrices in terms of their orthogonal group characters. As is to be expected, various cases have to be considered, depending on the parity of the dimensions of the matrices. {The reader is warned that in section 2 each of the letters ϕ_1, ϕ_2 is used to denote simultaneously three different things.}

H. K. Farahat (Sheffield)

5245:

Cohn, P. M. On the structure of sesquilateral division semigroups. Proc. London Math. Soc. (3) 8 (1958), 466–480.

By a d-semigroup S the author means a left simple semigroup without idempotent elements in which for $a\neq b \in S$ either $a=by$ or $b=ay$ holds for some $y \in S$.

It is proved that: Every proper homomorphic image of a d-semigroup is a group. Moreover there is a (uniquely determined up to isomorphism) "biggest" group G (called the group associated with S) such that G is a homomorphic image of S and every proper homomorphic image of S is also a homomorphic image of G. {Reviewer's note: This result (under more general suppositions) was also proved by Gluskin [Dokl. Akad. Nauk SSSR **102** (1955), 673–675; MR **17**, 237].}

In a d-semigroup write $a<b$ whenever $a=bz$ for some $z \in S$. The relation $<$ defines a total ordering of S, and S is dense-in-itself (i.e., given $a<b$, there is an $x \in S$ with $a<x<b$). Also, $xa<xb \Leftrightarrow a<b$ holds. Conclusions from these facts are derived.

A δ-semigroup is a left cancellation semigroup without idempotents in which for $a\neq b$ either $a=by$ or $b=ay$ holds for some $y \in S$. In a previous paper [J. London Math. Soc. **31** (1956), 181–191; MR **18**, 14] the author has shown that every δ-semigroup can be embedded in a d-semigroup. Some deeper details concerning this embedding are clarified and the homomorphism of a δ-semigroup contained in a d-semigroup S into the group G associated with S is studied.

Finally an interesting result is proved: A group G can be associated with a d-semigroup if and only if it can be left-ordered (i.e., a total ordering on G can be defined such that $a<b$ implies $xa<xb$ for all a, b, $x \in G$).

Št. Schwarz (Bratislava)

5246:

Šulka, Robert. On the isomorphism of topological groupoids. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 143–157. (Slovak. Russian and English summaries)

More about topological groupoids, their isomorphisms and realizations [See same Časopis. **5** (1955), 10–21; MR **16**, 997]. *E. Hewitt* (Seattle, Wash.)

TOPOLOGICAL GROUPS AND LIE THEORY

5247:

Pae, Mi-Soo. Uniform topology on a group. Kyungpook Math. J. 1 (1958), 43–47.

Let G be an abstract group which is also a T_1-space (but not necessarily a topological group). Let $\{V_a\}$ be a fundamental system of neighborhoods of the identity. Two types of uniform structures arising from this system are studied, and sufficient conditions are given that the topology determined by the uniform structure makes G a topological group. *P. S. Mostert* (New Orleans, La.)

5248:

Hochschild, G.; and Mostow, G. D. Representations and representative functions of Lie groups. Ann. of Math. (2) **66** (1957), 495–542.

Let F be an infinite field and let G be a group. The notion of a representative function on G is defined in the following two cases: (I) when G is a topological group and F is the real or complex numbers; and (II) when G is an algebraic group over F. In either case f, a mapping of G into F, is said to be a representative function if it arises, in the usual way, from a representation (assumed continuous in case I and rational in case II) of G on an F-linear space V.

The space of representative functions, $\mathbf{R}(G)$, forms an algebra on which the group operates, as automorphisms, in two (commuting) ways — left and right translations. A representative function is characterized in a larger class of functions (called admissible) in that its translates (left or right) span a finite-dimensional subspace. Finite-dimensional, left-translation stable, subspaces S of $\mathbf{R}(G)$ play an important role here. Not only are they representation spaces of G (operating as left translations), but they are also stable under the larger group A of all proper automorphisms of $\mathbf{R}(G)$. An automorphism of a subalgebra Q of $\mathbf{R}(G)$ which contains the constants and is stable under left and right translation is called proper if it leaves the constants fixed and commutes with right translations. It is shown then that if Q is stable under the operation $f\to f'$, where $f'(x)=f(x^{-1})$, then the proper automorphisms of Q are in a natural 1-1 correspondence with the homomorphisms of Q into F leaving the constants fixed. (When $Q=\mathbf{R}(G)$ one considers the set A_0 of such homomorphisms in the Tannaka theory. The approach presented here, that is, of replacing A_0 by A, has the advantage of providing A_0 with a group structure in a direct way.) It is also shown that if S is a finite-dimensional subspace of $\mathbf{R}(G)$ which is stable under left and right translations and G_S is the group of left translations of S, then its algebraic hull coincides with the restriction to S of the group of all proper automorphisms of the algebra generated by the constants and S. Among other results in this section it is proved that in Case (II),

assuming F is algebraically closed and Q is finitely generated (one of the authors informed the reviewer that this last condition was incorrectly omitted in the paper), every proper automorphism of Q is the restriction to Q of a proper automorphism of $\mathbf{R}(G)$. Also, if $G=HN$ is a semi-direct product (N normal), $\mathbf{R}(G)$ is canonically isomorphic to the tensor product $\mathbf{R}(H)\otimes\mathbf{R}(G)_N$, where $\mathbf{R}(G)_N$ is the restriction image of $\mathbf{R}(G)$ in $\mathbf{R}(N)$.

An element $f\in\mathbf{R}(G)$ is called semisimple if the representation on the space spanned by the left translates of f is semisimple. It is then proved that if F is of characteristic zero the set $\mathbf{R}_s(G)$ of semisimple elements is a subalgebra of $\mathbf{R}(G)$, and that if $\mathbf{R}(G)$ is finitely generated so is $\mathbf{R}_s(G)$.

Now, in Case (I), A is made into a topological group and, if F is the complex numbers, denote by B the closed subgroup of all $\alpha\in A$ such that α commutes with complex conjugation. In such a case the Tannaka theorem takes the following form: assume G is compact; the map which sends every element of G onto the corresponding left translation of $\mathbf{R}(G)$ is a topological isomorphism of G onto B. Also, in case G is compact, a new proof of a result of Nakayama is given establishing a Galois correspondence between all closed subgroups of G and all subalgebras S of $\mathbf{R}(G)$ which contain the constants and are stable under right translation and conjugation.

Now let G be a Lie group and let G_1 be the identity component. The following is proved. Assume G/G_1 is finite; let N be the analytic group corresponding to the radical of the commutator of the Lie algebra of G_1. Then N is nilpotent and simply connected, and coincides with the intersection of all kernels of semisimple representations of G. Furthermore, every representation of G induces a unipotent representation of N, and G/N has a faithful semisimple representation. The following result on the extendability of representations is also proved. Let $G=LN$ be a semi-direct product (N normal, simply connected). Let V be a unipotent representation space for N. Then V is an N-subspace of a representation space for G.

An universal complexification G^+ is defined. This group is such that there is a 1-1 correspondence between the representations of G by complex linear automorphisms and the complex representations of G^+. Furthermore, if r and r^+ so correspond, then r is the composite of r^+ with a canonical fixed homomorphism of G into G^+, and $r^+(G^+)$ is the smallest complex linear Lie group containing $r(G)$. A complex Lie group analogue of a theorem of Goto on faithful representations is proved. This implies that if G/G_1 is finite then G^+ has a faithful complex representation.

Now assume G/G_1 is finite. Analogously with the definition of A and B, let $\mathfrak{A}$ and $\mathfrak{B}$ be, respectively, the Lie algebra of all proper derivations of $\mathbf{R}(G)$ and the subalgebra of all such derivations which commute with complex conjugation. An exponential map is defined on $\mathfrak{A}$ sending $\mathfrak{A}$ into A and $\mathfrak{B}$ into B. Now let t denote the natural homomorphism (left translation) of G into B and let t^+ be the corresponding homomorphism of G^+ into A. It is shown that t^+ is a monomorphism and that t and t^+ have differentials $t\cdot$ and $t^+\cdot$ which, respectively, map the Lie algebras $\mathfrak{G}$ and $\mathfrak{G}^+$ of G and G^+ into $\mathfrak{B}$ and $\mathfrak{A}$. The following theorem contains some of the main results of the paper. It is a generalization of the Tannaka theorem. It asserts that the following are equivalent: (1) $\mathbf{R}(G)$ is finitely generated; (2) $t^+(G^+)=A$; (3) if K is the closure in G of the commutator subgroup of G_1, then G/K is compact; (4) for every representation r of G, $r^+(G^+)$ is an algebraic linear group; (5) $\mathfrak{A}$ is finite-dimensional; (6) $t\cdot(\mathfrak{G})=\mathfrak{B}$; (7) the identity component in B coincides with $t(G_1)$. A Lie group G satisfying any of the conditions of this theorem has been called by one of the authors (but not in this paper) a Tannaka group. If G is a Tannaka group it is also shown that G^+ can be identified with an algebraic complex linear group in such a way that every complex representation of G^+ becomes a rational representation, and the associated representative functions are in fact polynomial functions. It is also shown that if G is a Tannaka group and S is any finite-dimensional subspace of $\mathbf{R}(G)$ that is stable under left translations and the map $f\to f'$, and is such that the representation of G^+ on S is faithful, then the constants and the elements of S generate $\mathbf{R}(G)$. This result yields the following generalization of a theorem of Harish-Chandra (case when G is semisimple, connected). Let r be a representation of G such that r^+ is faithful. Then every representation of G belongs to the ring of representations generated by r and its dual.

B. Kostant (Berkeley, Calif.)

TOPOLOGICAL ALGEBRA

See also 5192, 5246.

5249:

Mal'cev, A. I. Free topological algebras. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 171–198. (Russian)

The theory of free topological groups suggests the study of a topological algebra, a universal algebra over a Hausdorff space. See Mat. Sb. N.S. **35(77)** (1954), 3–20 [MR **16**, 440]. For a given generating topological space X, a primitive class $\mathfrak{S}$, and a system S of defining relations, the existence and uniqueness of the topological algebra A in $\mathfrak{S}$ is discussed. Questions raised include: is A isomorphic to the abstract algebra in $\mathfrak{S}$ obtained from S and from X considered as a generating set?; does A contain X topologically as a subspace?; how is the topology in A described? The topological algebra A in $\mathfrak{S}$ is free if S is empty. The investigation of whether every topological algebra in $\mathfrak{S}$ is a continuous and open homomorphic image of a free topological algebra in $\mathfrak{S}$ is intimately connected with the question of whether $\mathfrak{S}$ is a class with commutative congruences. A free union of topological algebras is studied. The notion of a limit is analyzed from the point of view of a partial operation on infinitely many arguments. A transfinite procedure leading to the topology in A which is free relative to X begins with the so-called initial topology in A, a topology also of interest for its own sake. The nature of the topology in A for the important cases in which the generating topological space is bicompact or is locally compact is treated. A primitive class of algebras is called a β-class provided every free topological algebra in the class over a bicompact space X contains X topologically and is algebraically free over X. After establishing a convenient sufficient condition that a primitive class be a β-class, easy application shows that many classes of algebras are β-classes, such as, for example, the class of groups, the class of nilpotent groups having a given degree of nilpotency, the class of all (nonassociative) rings with characteristic zero, the class of Lie rings and the class of Boolean algebras.

R. A. Good (College Park, Md.)

FUNCTIONS OF REAL VARIABLES

See also 5195, 5463.

5250:

Karamata, J. Introduction à une théorie de la croissance des fonctions réelles. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) 1 (49) (1957), 295–302.

This is a lecture given before the Congress of Roumanian mathematicians at Bucharest, May-June, 1956. It refers to the work on the growth of real functions by P. du Bois-Reymond, G. H. Hardy, N. Bourbaki, and to the the recent (1956) thesis of M. Šteković. In the set of functions defined on a neighborhood of infinity the author considers the set F of "germs of functions" defined as equivalence classes with respect to the equivalence relation $f(\xi)=g(\xi)$ for every ξ in a neighborhood of infinity. He sets $f\preceq g$ or else $f\overset{\lambda}{\preceq}g$ if there exists a real number λ such that to every $\varepsilon>0$ there corresponds a $\xi(\varepsilon)$ such that $|f-\lambda g|\leq\varepsilon|g|$ for every $\xi>\xi(\varepsilon)$. If $\lambda\neq0$, then $f\overset{\lambda}{\preceq}g\Leftrightarrow g\overset{\mu}{\preceq}f$ with $\mu=1/\lambda$; in this case he calls f and g "asymptotically equivalent" and writes $f\asymp g$; this is an equivalence relation. On the other hand, if $\lambda=0$, then he sets $f\prec g$ or $g\succ f$, and he calls f "asymptotically smaller" than g. Then the set of "orders of growth" is the set quotient $F/\asymp$. On this basis the theory of growth of the functions f is then developed systematically.

A. Rosenthal (Lafayette, Ind.)

5251:

Tambs Lyche, R. Les quatre dérivées d'une fonction réelle et continue. Rev. Math. Pures Appl. 1 (1956), no. 3, 5–7.

The author gives an example (essentially the same that had been studied by B. L. van der Waerden [Math. Z. 32 (1930), 474–475]) of a real, continuous, nowhere differentiable function f with the following property: there exists a point set E which is everywhere dense and has the power of the continuum such that for $x\in E$ the four derivates are finite and $D^+f(x)>D_+f(x)$, $D^-f(x)>D_-f(x)$.

A. Rosenthal (Lafayette, Ind.)

5252:

Lozinskiĭ, S. M. On the indicatrix of Banach. Vestnik Leningrad. Univ. 13 (1958), no. 7, 70–87. (Russian. English summary)

Complete statements and proofs of results announced previously [Dokl. Akad. Nauk SSSR 60 (1948), 765–767; MR 10, 24].

E. Hewitt (Seattle, Wash.)

5253:

Ščedrova, N. S. Sequences of iterations involving functions. Grodnen. Gos. Ped. Inst. Uč. Zap. Ser. Mat. 2 (1957), 102–114. (Russian)

The author investigates the behaviour as $m\to\infty$ of a sequence $\{x_m\}_0^\infty$ defined by the relation

$$x_{sn+k}=f_k(x_{sn+k-1})\ (s=0,1,2,\cdots;\ k=1,2,\cdots),$$

where n is a fixed positive integer, x_0 is any real number, and the functions $f_k(x)$ are continuous and strictly monotonic (possibly in different sense) on $(-\infty,\infty)$. Her results are too elaborate for reproduction here, but they are related in a fairly obvious way to those for the subsequence $\{x_{sn}\}_{s=0}^\infty$, and hence to those for the case $n=1$, which are known [cf. W. Coppel, Proc. Cambridge Philos. Soc. 51 (1955), 41–43; MR 16, 577].

H. P. Mulholland (Exeter)

5254:

Osipov, P. M. On reducing a double curvilinear integral to a double plane integral. Dopovidi Akad. Nauk Ukrain. RSR 1958, 493–497. (Ukrainian. Russian and English summaries)

5255:

Łuszczki, Z. Application of generalized Bernstein polynomials to the proof of a certain theorem on mixed partial derivatives. Prace Mat. 2 (1958), 355–360. (Polish. Russian and English summaries)

Approximation by Bernstein polynomials in several variables may be used to show that two continuous mixed partial derivatives of a function $f(x_1,\cdots,x_r)$ are equal if they differ only in the order of differentiations.

G. G. Lorentz (Syracuse, N.Y.)

5256:

Lipiński, J. S. Sur les ensembles $\{f'(x)>a\}$. Fund. Math. 45 (1958), 254–260.

In a previous paper [Fund. Math. 42 (1955), 339–342; MR 17, 953] the author considered a function $f(x)$ (not necessarily continuous) with a finite or infinite derivative $f'(x)$ for every x, and answered two questions raised by Zahorski [Trans. Amer. Math. Soc. 69 (1950), 1–54; MR 12, 247]. In this paper he further restricts $f(x)$ to be continuous and obtains a further property of the sets $\{f'(x)>a\}$. This property, together with the condition that $\{f'(x)>a\}$ belongs to the class F_σ, is shown to imply the necessary condition M_2 of Zahorski.

U. S. Haslam-Jones (Oxford)

5257:

Sengupta, H. M.; and Lahiri, B. K. A note on derivatives of a function. Bull. Calcutta Math. Soc. 49 (1957), 189–191.

Let f be a real-valued function defined on $a\leq x\leq b$; let f be discontinuous on an everywhere dense set A and continuous on an everywhere dense set B. Then it is known that A is a set of the first category and, according to M. K. Fort, Jr. [Amer. Math. Monthly 58 (1951), 408–410], the set of points (if it exists) where f is differentiable is also a set of the first category. The authors prove the following generalization of Fort's result: Let E be the set where f is continuous and at least one of its four derivates D^+, D_+, D^-, D_- is infinite; then E is a residual set.

A. Rosenthal (Lafayette, Ind.)

5258:

Ciesielski, Z. On some inequalities. Prace Mat. 2 (1958), 361–367. (Polish. Russian and English summaries)

Theorem 1 of the paper proves th. 86, p. 72, of Hardy, Littlewood and Pólya ["Inequalities", 2nd ed., University Press, Cambridge, 1952; MR 13, 727] with the following additional conditions: $\phi'(x)$ is non-concave, $a_1\geq a_2\geq\cdots\geq a_n$, $a_i\in[0,b]$. The condition of H.L.P. that $\sum_{i=1}^n q_i=1$ is replaced by $q_i=\pm1$ and $\sum_{i=1}^p q_i\geq0$ for $p=1,2,\cdots,n$.

Theorem 2 establishes the inequality of th. 98, p. 80 of H.L.P. with the following modifications: ϕ is weakly non-concave (instead of non-concave), i.e.,

$$\phi\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)\leq\tfrac{1}{2}[\phi(x_1,y_1)+\phi(x_2,y_2)]$$

for any pair of points such that $x_1\geq x_2$, $y_1\geq y_2$; $D=[0,b_1]\times[0,b_2]$, $\partial\phi/\partial x$ and $\partial\phi/\partial y$ are continuous, weakly non-concave, nondecreasing in D (in the Russian and English summaries at the end of the paper "nondecreasing" is erroneously translated as "nonincreasing"); instead of $\sum q_i=1$, the q_i's must satisfy the same condition as in th. 1.

Theorem 3 proves that

$$1\leq\prod_{i=1}^{n}(1+b_i)^{a_i}\leq\exp\sum_{i=1}^{n}a_ib_i$$

if $b_i\geq b_{i+1}\geq 0$ and $\sum_{i=1}^{k}a_i\geq 0$ for $k=1, 2, \cdots, n$.

C. Masaitis (Havre de Grace, Md.)

5259:

Kalman, J. A. On the inequality of Ingham and Jessen. J. London Math. Soc. 33 (1958), 306–311.

The author attempted a generalization of Theorems 26, 203 in G. H. Hardy, J. E. Littlewood, and G. Pólya, "Inequalities" [2nd ed., University Press, Cambridge, 1952; MR 13, 727], which may be stated as follows: Let (M, μ) and (N, ν) be σ-finite measure spaces. Putting

$$\mathfrak{M}_r^{(x)}f(x)=\left\{\int_M f(x)^r\mu(dx)\right\}^{1/r},$$

$$\mathfrak{M}_s^{(y)}g(y)=\left\{\int_N g(y)^s\nu(dy)\right\}^{1/s}$$

for $0<r<s$ and measurable positive functions $f(x)$ on M and $g(y)$ on N, we have

(1) $$\mathfrak{M}_s^{(y)}\mathfrak{M}_r^{(x)}f(x, y)\leq\mathfrak{M}_r^{(x)}\mathfrak{M}_s^{(y)}f(x, y).$$

The author extended this inequality (1) to the case where r and s are measurable functions, respectively, on M and N such that $0<r(x)\leq s(y)$ for all x in M and y in N, putting

$$\mathfrak{M}_r^{(x)}f(x)=\inf\{1/k: m(kf)\leq 1\},\ m(f)=\int_M f(x)^{r(x)}\mu(dx),$$

$$\mathfrak{M}_s^{(y)}g(y)=\inf\{1/k: n(kg)\leq 1\},\ n(g)=\int_M g(y)^{s(y)}\nu(dy).$$

Here m and n are modulars, and $\mathfrak{M}_r^{(x)}f(x)$ and $\mathfrak{M}_s^{(y)}g(y)$ are "second norms", respectively, by m and n [cf. H. Nakano, "Modulared semi-ordered linear spaces", Maruzen, Tokyo, 1950; MR 12, 420]. {The reviewer wants to remark that further extension of (1) will be considered for more general modulars m and n, if $r\leq s$, for the "upper exponent" r by m and the "lower exponent" s by n [cf. S. Yamamuro, J. Fac. Sci. Hokkaido Univ. Ser. I. 12 (1953), 211–253; MR 16, 50].} *H. Nakano* (Sapporo)

5260:

Sengupta, H. M.; and Lahiri, B. K. A note on implicit functions. Bull. Calcutta Math. Soc. 49 (1957), 79–82.

Generalizing a result of A. S. Besicovitch [see G. H. Hardy, A course of pure mathematics, 10th ed., Univ. Press, Cambridge, 1952; MR 14, 145; p. 203] the authors prove the following theorem: (1) Let $f(x, y)$ be a continuous function of x in a neighborhood of a, $a-\varepsilon_1\leq x\leq a+\varepsilon_1$, for an everywhere dense set of values of y in a neighborhood of b, $b-\varepsilon_2\leq x\leq b+\varepsilon_2$, and be a continuous function of y in the neighborhood of b, for all values of x in the neighborhood of a; (2) let $f(a, b)=0$; (3) let $f(x, y)$ be a strictly increasing (or decreasing) function of y in the neighborhood of b for an everywhere dense set S of values of x in the neighborhood of a including a; (4) let any zero (ξ, η) of $f(x, y)$ in the neighborhood of (a, b), $a-\varepsilon_1\leq x\leq a+\varepsilon_1$, $b-\varepsilon_2\leq y\leq b+\varepsilon_2$, on $x=\xi$, $\xi\in C(S)$ (where $C(S)$ designates the complementary set of S with respect to the neighborhood of a) be the limit point of two sequences of points $(\xi, \eta+\Delta_n)$, $\Delta_n>0$, $\Delta_n\to 0$, $(\xi, \eta-\Delta_n')$, $\Delta_n'>0$, $\Delta_n'\to 0$, such that $f(\xi, \eta+\Delta_n)\cdot f(\xi, \eta-\Delta_n')<0$ $(n=1, 2, 3, \cdots)$. Then: (α) there exists a unique function $y=\varphi(x)$ which satisfies the equation $f(x, y)=0$ in a neighborhood of a; (β) φ is continuous in this neighborhood of a.

A. Rosenthal (Lafayette, Ind.)

5261:

Soble, A. B. Abstract structure of inequalities. Math. Mag. 31 (1957/58), 179–184.

Several remarks about order relations between real numbers or real-valued functions. Example: if $F\to F_*$ takes an interval strictly monotonically onto $(0, \infty)$, then the operation $F\otimes G$ such that $(F\otimes G)_*=F_*+G_*$ has the property that $F\otimes G\geq F$. *C. Davis* (Providence, R.I.)

5262:

Sendov, Blagovest. On the problem of expanding regularly monotone functions in Gončarov series. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 450–453. (Russian)

A new proof is sketched and generalizations are announced for a theorem announced previously [Dokl. Akad. Nauk SSSR 110 (1956), 27–30; MR 18, 476].

E. Hewitt (Seattle, Wash.)

5263:

Kosmák, Ladislav. Une note sur les fonctions convexes. Czechoslovak Math. J. 6(81) (1956), 420–425. (Russian summary)

Let f be a function continuous on the segment $\langle a, b\rangle$ and let $P_{\langle a,b\rangle}$ be the polynomial function of second degree which is the best approximation (supremum norm) to f on $\langle a, b\rangle$. Let

$$E_{\langle a,b\rangle}=\max_{a\leq x\leq b}|f(x)-P_{\langle a,b\rangle}|.$$

If $f\neq P_{\langle a,b\rangle}$ on $\langle a, b\rangle$ let $m_{\langle a,b\rangle}$ be the maximum number of elements in a finite sequence

$$a\leq\xi_1<\xi_2<\cdots<\xi_k$$

such that $|f(\xi_i)-P_{\langle a,b\rangle}(\xi_i)|=E_{\langle a,b\rangle}$ and $f(\xi_i)-P_{\langle a,b\rangle}(\xi_i)$ alternates in sign with i. If $f=P_{\langle a,b\rangle}$ in $\langle a, b\rangle$ let $m_{\langle a,b\rangle}$ be 3. The author announces the following theorem. The function $f(x)$ defined and continuous on $\langle a, b\rangle$ is convex if and only if for every subsegment, $\langle\alpha, \beta\rangle$, of $\langle a, b\rangle$ one has

$$E_{\langle\alpha,\beta\rangle}\leq\frac{(\beta-\alpha)^2}{2(m_{\langle\alpha,\beta\rangle}-1)^2}c_{\langle\alpha,\beta\rangle},$$

where $c_{\langle\alpha,\beta\rangle}$ is the coefficient of x^2 in $P_{\langle\alpha,\beta\rangle}$.

{In making the direct proof the author assumes that at a certain point x_1 of maximum distance the derivative of f exists and equals $P_{\langle a,b\rangle}'(x_1)$. The reviewer has not verified that this derivative exists although the author states that it is easily seen. It seems likely that a modification of the proof may be made even if the derivative does not exist.} *P. C. Hammer* (Madison, Wis.)

MEASURE AND INTEGRATION

See also 5252.

5264:

Vinogradov, I. M. A multiple integral. Izv. Akad. Nauk SSSR. Ser. Mat. 22 (1958), 577–584. (Russian)

An improved estimate is given for the integral

$$\int_0^1\cdots\int_0^1|T_0|^{2b}d\alpha_n\cdots d\alpha_1,$$

where

$$T_0=\sum_{0<x\leq p_0}\exp 2\pi i(\alpha_nx^n+\cdots+\alpha_1x).$$

Author's summary

5265:

Dževarešelšvili, A. G. On integration of the product of two functions. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 35–51. (Russian)

A continuous function $F(x, y)$, defined on $\{[a, b]\cdot[c, d]\}$, is said to be A_x on $P\subset[a, b]$ [or A_y on $Q\subset[c, d]$] if $P=\sum P_k$ and F is AC (absolutely continuous in the sense of Čelidze) on each $P_k[c, d]$. $F(x, y)$ is said to satisfy condition I_x on $[a, b]$ if any perfect set E contains a portion $P=[\alpha, \beta]E$ such that for any $x', x''\in P$ the inequality $|F(x', y)-F(x'', y)|\leq|\psi(x'')-\psi(x')|$ holds uniformly in $y\in[c, d]$, where $\psi(x)$ is a fixed AC function on $[a, b]$. The author proves, among others, the following results. Theorem 3: If

$$F(x, y)=\int_a^x\int_c^y f(t, s)dtds$$

is A_x on $[a, b]$, then for any $\{[\alpha, \beta]\cdot[\gamma, \delta]\}\subset[a, b]\cdot[c, d]$ it is true that

$$\int_\alpha^\beta\int_\gamma^\delta f(t, s)dtds=\int_\alpha^\beta dt\int_\gamma^\delta f(t, s)ds.$$

Theorem 4: If $F(x, y)$ is A_x on $[a, b]$ and if $F(x, y_0)$ is ACG on $[a, b]$, then F satisfies condition I_x on $[a, b]$. Theorem 5: If

$$F(x, y)=\int_0^x\int_0^y f(t, s)dtds$$

is I_y on $[c, d]$, and if $\varphi(x)$ is AC on $[a, b]$, then $f(x, y)\varphi(x)$ is integrable on any $\{[\alpha, \beta]\cdot[\gamma, \delta]\}$ and

$$\int_\alpha^\beta\int_\gamma^\delta f(t, s)\varphi(t)dtds=(F(\alpha, \gamma)-F(\alpha, \delta))\varphi(\alpha)$$
$$+(F(\beta, \delta)-F(\beta, \gamma))\varphi(\beta)-\int_\alpha^\beta[F(x, \delta)-F(x, \gamma)]\varphi'(x)dx.$$

If, in addition, F is A_y on $[c, d]$ and $\psi(y)$ is AC on $[c, d]$, then $f(t, s)\varphi(t)\psi(s)$ is integrable on $[a, b]\cdot[c, d]$ and

$$\int_\alpha^\beta\int_\gamma^\delta f(t, s)\varphi(t)\psi(s)dtds=\int_\alpha^\beta dt\int_\gamma^\delta f(t, s)\varphi(t)\psi(s)ds=$$
$$\int_\gamma^\delta ds\int_\alpha^\beta f(t, s)\varphi(t)\psi(s)dt.$$

M. Cotlar (Buenos Aires)

5266:

Bertolini, Fernando. Le funzioni additive nella teoria algebrica della misura. Ann. Scuola Norm. Sup. Pisa (3) 12 (1958), 155–162.

5267:

Kvačko, M. E. Measurable mappings of spaces. Vestnik Leningrad. Univ. 13 (1958), no. 13, 87–101. (Russian. English summary)

Egorov's theorem and Luzin's theorem on "almost continuity" of measurable functions are proved for mappings into separable metric spaces. Weak convergence of induced measures on the image spaces is also discussed. *E. Hewitt* (Seattle, Wash.)

5268:

Kononov, V. A. Topological properties of a dynamic system with invariant measure. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 1038–1041. (Ukrainian. Russian and English summaries)

"This article deals with the problem of behaviour of the trajectory in an abstract dynamic system $f(p, t)$ to within a certain measure ν. The notion of the limiting trajectory, including, in the case of a plane, isolated cycles, is introduced in this article.

Then assuming the existence of the invariant measure μ represented by

$$\mu A=\int_A M(p)d\nu,$$

a criterion enabling us to define the limiting trajectory is given by the function $M(p)$." *Author's summary*

5269:

Hüsser, Rudolf; und Nef, Walter. Ein Beitrag zum Stieltjesschen Integralbegriff. Mitt. Verein. Schweiz. Versich.-Math. **58** (1958), 167–175.

A Stieltjes integral is defined on a subset B of n-dimensional Euclidean space relative to an additive set function $F(X)$ on an algebra of subsets $\mathfrak{M}$. It is assumed that B is bounded and for every $\varepsilon>0$ can be covered by a finite number of disjoint subsets in $\mathfrak{M}$ each of diameter less than ε. Then

$$\int_B\varphi dF=\lim_{D\to0}\sum_{m=1}^n\varphi(x_m)F(X_m),$$

where $\sum X_m$ is a finite covering of B by disjoint sets in $\mathfrak{M}$, x_m belongs to the intersection of X_m and B, and diam $X_m\leq D$. It is proved that this integral exists if φ is continuous and B has the following property: for every $\varepsilon>0$ there exists D_1 such that if $Z_1, \cdots, Z_r$ have diam $Z_k<D_1$ and each contains elements of B and the complement of B, and if, further, $Y_1, \cdots, Y_r$ are in $\mathfrak{M}$, disjoint, and $Y_k\subset Z_k$ for all k, then $\sum_{k=1}^r|F(Y_k)|<\varepsilon$. The integral definition is related to that proposed by McShane and Botts [Duke Math. J. **19** (1952), 293–302; MR **13**, 924]. Application is made to some problems in actuarial mathematics. *T. H. Hildebrandt* (Ann Arbor, Mich.)

5270:

Matthes, Klaus. Über eine Verallgemeinerung des Lebesgueschen Integralbegriffes. II. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe 6 (1956/57), 221–236. (Russian, English and French summaries)

Die Arbeit befaßt sich mit σ-Homomorphismen Boolescher σ-Algebren als Verallgemeinerungen der die gewöhnlichen "Punktfunktionen" beschreibenden Spektralscharen, und mit Maßen, deren Werte positive Elemente eines bedingt vollständigen Vektorverbandes V sind. Aus einem Satz über die Ausdehnung von σ-Homomorphismen, die zunächst nur in einer Algebra oder Halbalgebra definiert sind, ergibt sich der folgende Satz, in dem I einen nicht leeren Parameterbereich und B_I die von den Produkten Borelscher Mengen in R^1 mit nur endlich vielen von R^1 verschiedenen Faktoren erzeugte σ-Algebra von Teilmengen von R^I bedeutet: Zu jeder Familie $(\sigma_\iota)_{\iota\in I}$ von Spektralscharen in einer σ-Algebra B gibt es einen und nur einen σ-Homomorphismus φ von B_I in B, sodaß für jedes $\iota\in I$ und $\lambda\in R^1$ gilt $\varphi(\{x: x_\iota<\lambda\})=\sigma_\iota(\lambda)$. Dies gestattet, den Begriff einer Funktion einer Familie von Spektralscharen zu definieren als Spektralschar $(\phi((\sigma_\iota)_{\iota\in I}))(\lambda)=\varphi(\varphi^{-1}(]-\infty, \lambda[))$, wenn ϕ eine B_I-meßbare reelle Funktion ist; auch erhält man als Spezialfall den Loomisschen Darstellungssatz. Es folgt ein Satz über die Ausdehnung eines Maßes mit Werten in V, das zunächst nur in einer Halbalgebra definiert ist, und als unmittelbare Anwendung die folgende Feststellung: Zu jeder endlich additiven strikt positiven Abbildung einer Algebra B in V, die einer gewissen σ-Endlichkeitsbedingung genügt, gibt es einen Isomorphismus ψ von B in eine σ-Algebra B', die von $\psi(B)$ erzeugt wird, sodaß $\mu\circ\psi^{-1}$ zu einem Maß B' ausgedehnt werden kann. Weiter werden "Verbindungsmaße" betrachtet, die

Verallgemeinerungen der gemeinsamen Verteilungen einer Familie zufälliger Variabler sind: wird eine Familie $(B_t)_{t\in I}$ von σ-Algebren durch eine entsprechende Familie $(\varphi_t)_{t\in I}$ von σ-Homomorphismen in ein und dieselbe σ-Algebra B abgebildet, auf der ein Maß μ mit Werten in V vorliegt, so sei

$$\tilde{\mu}_{t_1,\cdots,t_n}(a_1,\cdots,a_n)=\mu(\varphi_{t_1}(a_1)\wedge\cdots\wedge\varphi_{t_n}(a_n)).$$

Bewiesen wird bei gegebenen Verbindungsmaßen ein Eindeutigkeitssatz "bis auf Isomorphie" für die von allen $\varphi_t(a_t)$ erzeugte σ-Algebra B' nebst den φ_t und der Einschränkung von μ auf B', der insbesondere einen Eindeutigkeitssatz für die oben aufgetretene σ-Algebra B' enthält, und ein entsprechender Existenzsatz, wenn man allein von den durch gewisse Eigenschaften erklärten Verbindungsmaßen ausgeht.

K. Krickeberg (Heidelberg)

5271:

Slepian, Paul. Theory of Lebesgue area of continuous maps of 2-manifolds into n-space. Ann. of Math. (2) **68** (1958), 669–689.

The following basic inequality between the Lebesgue area of a surface and the sum of the areas of its projections is established: If f is a continuous function on a finitely triangulable subset of a 2-dimensional manifold into n-space $E_n(x_1, x_2, \cdots, x_n)$, then the Lebesgue area of f does not exceed the sum of the Lebesgue areas of the projections of f into the 2-dimensional subspaces $E_2(x_i, x_j)$, $1\leq i<j\leq n$, of E_n. This had been proven earlier by Federer for the case where the domain of f is a subset of the plane. The proof is accomplished by treating first the case where f is light and the domain of f is a compact 2-manifold. Earlier results of the author then enable one to limit consideration to the case of a compact 2-manifold as the domain of f. Then by factoring f into its monotone and light factors and employing the Roberts-Steenrod result on monotone maps applied to compact 2-manifolds, it turns out that with suitable small neighborhoods deleted from the middle space each proper cyclic element of the components remaining is homeomorphic with a subset of a compact 2-manifold. An approximation technique then enables one to complete the argument.

G. T. Whyburn (Charlottesville, Va.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 5315, 5367, 5426, 5463, 5470, 5481.

5272:

Usmanov, N. K. Linear boundary problems for generalized analytic functions. Latvijas PSR Zinātnu Akad. Vēstis 8(73) (1953), 115–121. (Russian)

5273:

Storvick, D. A. On pseudo-analytic functions. Nagoya Math. J. **12** (1957), 131–138.

A function $f(z)=u(x, y)+iv(x, y)$ is called (by this author) pseudo-analytic on a domain D if it is interior in the sense of Stoilow and if (i) u_x, u_y, v_x, v_y exist and are continuous in D, (ii) the Jacobian is positive except for a countable set. The pseudo-analytic functions $f(z)$ which have bounded dilatation quotient on $|z|<1$ and for which $\lim_{r\to1}|f(re^{i\theta})|=1$ for all θ except for a possible set of logarithmic capacity zero form the class (U^*). The author proves: Let $w=f(z)$ be in (U^*) on $|z|<1$ and suppose it omits the value 0. Then $f(z)$ is constant or $f(z)$ admits either 0 or ∞ as an asymptotic value. Finally, e.g., for the non-constant functions in (U^*) such that $|f(z)|<1$ on $|z|<1$, the author shows that the set of values of w in $|w|<1$ such that $f(z)=w$ has no solution in $|z|<1$ is of capacity zero.

C. J. Titus (Berkeley, Calif.)

5274:

Haefeli, Hans Georg. Konformitätsbedingungen bei vierdimensionalen Abbildungen. Ann. Acad. Sci. Fenn. Ser. A. I, no. **250/11** (1958), 8 pp.

The author observes that 3-vectors and 4-vectors in the Grassmann algebra over the real space E_4, and various orthogonality relations among vectors, can be conveniently expressed using the quaternion algebra. He then describes quaternionically various properties of a mapping W of E_4 into E_4: that it belong to Fueter's class of 'rechtanalytisch' quaternion functions; that it be analytic in two suitably chosen complex variables; and, assuming W linear, that it map hyperspheres onto certain ellipsoids.

F. D. Quigley (New Haven, Conn.)

5275:

Daïovitch, Voïn. Sur l'existence des valeurs limites de la résultante des fonctions appartenant à la classe H_δ, $\delta>1$. Bull. Soc. Math. Phys. Serbie **8** (1956), 23–28. (Serbo-Croatian summary)

Let

$$f(z)=\sum_0^\infty a_nz^n,\quad g(z)=\sum_0^\infty b_nz^n.$$

Then the Hadamard resultant

$$F(z)=\sum_0^\infty a_nb_nz^n$$

is given by the integral

$$F(z)=\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta})g\left(\frac{z}{R}\cdot e^{-i\theta}\right)d\theta,$$

where $|z|<R<1$. If now $1/\delta+1/\delta'=1$, and

$$\int_4^{2\pi}|f(re^{i\theta})|^\delta d\theta \text{ and } \int_0^{2\pi}|g(re^{i\theta})|^{\delta'}d\theta$$

are bounded as $r\to1$, then Hölder's inequality shows that $F(z)$ is bounded in $|z|<1$. This is the author's theorem, but his proof is more complicated.

W. K. Hayman (London)

5276:

Rajagopal, C. T. On an absolute constant for a class of power series. Math. Scand. **5** (1957), 267–270.

If $f(z)=\sum c_nz^n$ in $|z|<1$, if $c_0\geq0$ (misprinted as $c_n\geq0$) and $\Re f(z)<1$ in $|z|<1$, and if $g(r)=\sum|c_n|r^n$, then $g(\frac{1}{3})\leq1$ and $\inf_{r<1}g(r)/r\leq3$; the constant 3 is the best possible.

G. Piranian (Ann Arbor, Mich.)

5277:

Zmorovič, V. A. On the generalisation of Schwarz's integral formula on n-connected circular domains. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 489–492. (Ukrainian. Russian and English summaries)

In this paper the author establishes a new form of the generalisation of Schwarz's well-known integral formula on n-connected circular domains which is more convenient for various applications than H. Meschkowski's [Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys. no. **166** (1954); MR **15**, 695]. Using this formula the author establishes the structural formulas for three important

classes of uniform regular functions in n-connected circular domains, which cannot be done by Meschkowski's formula. *Author's summary*

5278:

Sakashita, Hideo. On the conformal mapping of nearly circular domains. Bull. Kyoto Gakugei Univ. Ser. B, no. 8 (1956), 10–14.

Let the simply connected domain G_1, which contains $|z|<p$ and is contained in $|z|<1$, be represented conformally on the simply connected G_2, which contains $|w|<q$ and is contained in $|w|<1$, by the function $w=f(z)$. If $f(0)=0$ and $f'(0)>0$, the author gives the inequality

$$|f(z)-z|\leq\theta\{(pR)^2+4p\sin^2 A\}^{\frac{1}{2}},$$

where

$$R=\max\{(1-p)/p,\ 1-q\},$$

$$A=\min\{\tfrac{1}{2}\pi,\ (\tfrac{1}{2}\pi)(-\log pq)\log(1+\theta)/(1-\theta)\}.$$

With $q=1$ this is shown to be more precise than one given by Ostrowski in this case [Jber. Deutsch. Math. Verein. **54** (1950), 78–81]. In the general case M. Müller [Math. Z. **43** (1938), 628–636] gave the inequality which arises with $q=1$ in the formula for A. The present author shows this to be untenable by construction of a special example. *A. J. Macintyre* (Cincinnati, Ohio)

5279:

Legendre, Robert. Fonctions Fuchsiennes symétriques de deuxième famille. C. R. Acad. Sci. Paris **247** (1958), 770–772.

The author considers a symmetric fuchsian half-polygon with n vertices on the real τ-axis, subject to the very convenient condition that all functions $Q(s)$ rationally defined on the polygon are single-valued in a single function $s(\tau)$, normalized by $s(1)=\infty$, $s(0)=1$, and $s(\infty)=0$. The author lets P_i denote the Eisenstein series of dimension -4 for the vertices equivalent to the vertex τ_i and he constructs identities relating $P_i(\tau)$, $s(\tau)$, and the zeros and poles of $Q(s)$, as well as $ds/d\tau$. Not all definitions and proofs are given, but the results are plausible from the general theory as found in Poincaré's famous memoirs [Acta Math. **1** (1882), 1–62, 193–294].

{The reviewer notes the absence of bibliography and the abruptness of the termination leading to the possible presumption that the last part of the manuscript may have been lost accidentally by the printer.}

H. Cohn (Tuscon, Ariz.)

5280:

Shah, S. M.; and Singh, S. K. Note on a step function. Quart. J. Math. Oxford Ser. (2) **9** (1958), 63–67.

The authors establish the relation

$$\limsup N(r)/n(r)\log r\leq 1-\lambda/\rho$$

(λ the lower order, ρ the order, $n(r)$ the number of zeros of an entire function).

They rely on a discussion of increasing step functions and elucidate various extreme cases by the construction of examples. *A. J. Macintyre* (Cincinnati, Ohio)

5281:

Edrei, Albert. Gap and density theorems for entire functions. Scripta Math. **23** (1957), 117–141 (1958).

After an introduction summarizing the known results, the author develops the work of the reviewer, which took no account of order [Proc. London Math. Soc. (3) **2** (1952), 286–296; MR **14**, 259], as follows.

If the entire function $F(z)=\sum c_n z^{\lambda_n}$ is of finite order ρ and is bounded on the positive real axis $z>0$, then $\frac{1}{2}\rho\leq\liminf(\sum 1/\lambda_n)/\log x$ (summation over $\lambda_n\leq x$). A function corresponding to a given $\{\lambda_n\}$ is constructed to show that the constant $\frac{1}{2}\rho$ is best possible.

Similar results are obtained for the sequence of sign changes in the coefficients of the power series of real entire functions. (The reviewer's work quoted is also extended to this situation.)

Maximum and minimum logarithmic densities for sequences of integers are defined and discussed. (A logarithmic density can exist for sequences unmeasurable in the sense of Pólya.) A group of theorems is given relating these to the order of the entire function $F(z)=\sum c_n z^{\lambda_n}$ in an angle. These are similar but not equivalent to corresponding theorems of Pólya in which the type of $F(z)$ is in question. A similar group of theorems involves changes of sign in the case of real functions.

The methods used depend on interpolating the coefficients in terms of other entire functions and relating the growth properties of these to the distribution of their zeros. *A. J. Macintyre* (Cincinnati, Ohio)

5282:

Rahman, Q. I. On the lower order of entire functions defined by Dirichlet series. Quart. J. Math. Oxford Ser. (2) **7** (1956), 96–99.

Let the Dirichlet series

$$f(s)=\textstyle\sum_1^\infty a_n e^{s\lambda_n}\ (\lambda_{n+1}>\lambda_n\to\infty,\ \lambda_1\geq 0,\ s=\sigma+it)$$

converge absolutely in the whole plane, let

$$M(\sigma)=\max f(\sigma+it)\ (-\infty<t<\infty),$$

$$\rho=\limsup_{\sigma\to\infty}\frac{\log\log M(\sigma)}{\sigma},\ \lambda=\liminf_{\sigma\to\infty}\frac{\log\log M(\sigma)}{\sigma}.$$

It is known that, if $D=\limsup((\log n)/\lambda_n)$ $(n\to\infty)$ is finite,

$$\limsup(\lambda_n\log\lambda_n/\log|a_n|^{-1})=\rho\ \ (n\to\infty).$$

The corresponding result for λ need not hold; it does not, for instance, for series with gaps or latent gaps. It is true, however, if $\log\lambda_n\sim\log\lambda_{n+1}$, $D<\infty$, and $\log|a_n/a_{n+1}|(\lambda_{n+1}-\lambda_n)^{-1}$ is a non-decreasing function of n for $n>n_0$, as the author shows, giving an even more detailed result. The conditions do not imply that $f(s)$ be of regular growth; and he constructs a function $f(s)$ such that $a_n>0$, $\log\lambda_n\sim\log\lambda_{n+1}$, $D=0$, $(\log a_n/a_{n+1})(\lambda_{n+1}-\lambda_n)^{-1}$ is a steadily increasing function of n, and yet $\rho>\lambda$.

H. Kober (Birmingham)

5283:

Balk, M. B. A theorem on entire functions. Moskov. Oblast. Pedagog. Inst. Uč. Zap. **57** (1957), 51–53. (Russian)

The author gives an elementary proof of the theorem [Čebotarev and Meĭman, Trudy Mat. Inst. Steklov. **26** (1949); MR **11**, 509; pp. 162–163] that if $v(z)$ is an entire function with real coefficients that has nonnegative imaginary part for $y>0$, then $v(z)$ is linear. Proof: Let $v(z)=\sum_{n=0}^\infty c_n z^n$. Then $c_0\geq 0$, and since v takes conjugate values at conjugate points, the imaginary part of $yv(z)$ is always nonnegative. Putting $z=re^{i\phi}$ we find

$$I\equiv c_1+c_1 r\cos\phi+\sum_{n=2}^\infty(c_{n+1}r^{n+1}-c_{n-1}r^{n-1})\cos n\phi\geq 0.$$

Then

$$|c_{n+1}r^{n+1}-c_{n-1}r^{n-1}|=2\pi^{-1}\left|\int_0^\pi I\cos n\phi d\phi\right|$$
$$\leqq 2\pi^{-1}\int_0^\pi I d\phi=2c_1.$$

Since r can be arbitrarily large this shows inductively that $c_n=0$ for $n\geqq 2$. *R. P. Boas, Jr.* (Evanston, Ill.)

5284:

Macintyre, Sheila Scott. μ-transforms and interpolation series: Abel's series. Proc. London Math. Soc. (3) **8** (1958), 481–492.

A. Gelfond [Mat. Sb. N.S. **4** (1938), 115–147] used the Pólya integral representation for an entire function $F(z)$ of exponential type to discuss the representation of $F(z)$ by its Abel series

$$F(0)+\sum_1^\infty \frac{z(z-n)^{n-1}}{n!}F^{(n)}(n).$$

A similar approach was used by Schmidli [thesis, Zurich, 1942; MR **4**, 39], and by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 288–292; Trans. Amer. Math. Soc. **64** (1948), 283–298; MR **9**, 232; **10**, 693], who also discussed more general integral representations and summability of the series. As a result of these investigations, it is known that the Abel series converges if $F(z)$ obeys the growth restriction

$$|F(re^{i\theta})|\leqq O(1)\psi(r)\exp[rH(\theta)],$$

where $\int \psi(r)dr<\infty$ and $H(\theta)$ is a certain positive function whose minimum value is .278; a necessary condition for convergence is the same condition with $\psi(r)=r^2$. For Mittag-Leffler summability, a sufficient condition is $|F(re^{i\theta})|=O(1)\exp(\tau r)$ with $\tau<1$. In the present paper, the author proves convergence for a slightly different class of entire functions F neither contained in nor containing the previously known class. Her method is based upon the use of a modified integral representation appropriate to the problem, and similar to her approach to the Newton series [Proc. London Math. Soc. (3) **4** (1954), 385–401; MR **16**, 687].

R. C. Buck (Stanford, Calif.)

5285:

Shankar, Hari. A note on entire and meromorphic functions. Publ. Math. Debrecen **5** (1958), 213–216.

Let $f(z)$ be an entire function with $|f(0)|=1$; let $\mathfrak{M}(r)$ be its minimum modulus; other notations as usual. The author improves results of S. K. Singh [J. Univ. Bombay (N.S.) **20** (1952), part 5, Sect. A, 1–7; Publ. Math. Debrecen **3** (1953), 1–8; MR **14**, 33; **15**, 786], as follows: If $0<r<R$, then

$$\exp(T(R))/\mathfrak{M}(r)\geqq(R/r)^{n(r)},$$
$$\mathfrak{M}(R)/\exp(T(r))\geqq(R/r)^{n(R)},$$
$$\exp(T(R))/\mathfrak{M}(r)\geqq(R/r)^{N(R)/\log R}.$$

He also proves that for any nonconstant meromorphic function, $T(\alpha r)-T(r)$ increases if $0<\alpha<1$.

R. P. Boas, Jr. (Evanston, Ill.)

5286:

Chuang, Chi-tai. Un théorème général sur les fonctions holomorphes dans le cercle unité et ses applications. I, II. Sci. Sinica **6** (1957), 569–621, 757–831.

"Le but de ce travail est d'établir, par la méthode de Wiman-Valiron-Bloch sous une forme simplifiée donnée par M. Macintyre, un théorème général sur les fonctions holomorphes dans le cercle unité, théorème qui complète et précise un résultat antérieur, et de donner ses applications.

Le premier chapitre est consacré à la démonstration d'un théorème préliminaire sur les fonctions convexes croissantes qui est une généralisation d'un lemme de M. Macintyre. En m'appuyant sur ce théorème préliminaire j'établis dans le second chapitre un théorème général sur les fonctions holomorphes dans le cercle unité dont l'énoncé se simplifie dans des cas particuliers.

Dans le troisième chapitre, je m'occupe de l'étude des fonctions holomorphes dans le cercle unité ne prenant pas la valeur zéro et de leurs dérivées. Je parviens à certains résultats qui comprennent en particulier les propositions suivantes dans lesquelles $M(r, F)$ désigne le maximum de $|F(z)|$ pour $|z|=r$ et $\operatorname{Arg} Z$ désigne l'argument réduit de Z.

Soient $F(z)$ une fonction holomorphe pour $|z|<1$ ne prenant pas la valeur zéro et $K>1$ un nombre. Supposons qu'il existe une valeur $0<r_0<1$ telle que

$$\log M(r_0, F)\geqq c(1+\log K+\log^+|F(0)|)e^a \quad (a=-\lambda/\log r_0).$$

Dans ces conditions, à tout nombre $0\leqq\omega<2\pi$ correspond un domaine D intérieur au cercle $|z|<1$ tel que $F(z)$ soit univalente dans D et le représente sur la couronne fendue

$$K^{-1}<|Z|<K,\quad 0<\operatorname{Arg}(e^{-i\omega}Z)<2\pi.$$

c et λ' sont deux constantes positives numériques.

Soient $F(z)$ une fonction holomorphe pour $|z|<1$ ne prenant pas la valeur zéro, ν un entier positif et $K>1$ un nombre. Supposons qu'il existe une valeur $0<r_0<1$ telle que

$$\log M(r_0, F)\geqq c_\nu\left(1+\log K+\log^+|F(0)|+\log\frac{1}{1-r_0}\right)e^a$$
$$(a=-\lambda'/\log r_0).$$

Dans ces conditions, à tout nombre $0\leqq\omega<2\pi$ correspond un domaine D intérieur au cercle $|z|<1$ tel que $F^{(\nu)}(z)$ soit univalente dans D et le représente sur la couronne fendue

$$K^{-1}<|Z|<K,\quad 0<\operatorname{Arg}(e^{-i\omega}Z)<2\pi.$$

c_ν est un nombre positif ne dépendant que de ν.

Il est clair que la première proposition contient le théorème de Schottky et la deuxième proposition contient un théorème analogue sur les fonctions holomorphes dans le cercle unité ne prenant pas la valeur 0 et dont la dérivée d'ordre ν ne prend pas la valeur 1. Je parviens aussi à des résultats sur les domaines d'univalence communs à une fonction holomorphe dans le cercle unité ne prenant pas la valeur zéro et à ses dérivées successives jusqu'à un certain ordre.

Je passe ensuite à l'étude des fonctions méromorphes dans le cercle unité prenant un nombre fini de fois les valeurs zéro et infini, elle fait l'objet du quatrième chapitre. Les résultats généralisent ceux obtenus dans le chapitre précédent.

Dans le cinquième chapitre, je montre que l'on peut obtenir des résultats plus précis dans un certain sens. Par exemple, la première proposition énoncée ci-dessus peut être précisée en ce qui concerne le diamètre du domaine D; on peut exiger que ce diamètre est inférieur à un nombre positif arbitraire donné d'avance.

Enfin dans le sixième chapitre, j'obtiens quelques résultats sur les domaines de remplissage communs à une fonction méromorphe dans le cercle unité prenant un nombre fini de fois les valeurs zéro et infini et à ses dérivées successives jusqu'à un certain ordre et sur les "flat regions" communs à une telle fonction et à ses dérivées successives jusqu'à un certain ordre.

Une partie des résultats exposés dans ce Mémoire a fait l'objet d'une note insérée dans Science Record (N.S.) 1 (1957), no. 1, 37–40 [MR 20 #1752]." (Author's introduction) *A. J. Macintyre* (Cincinnati, Ohio)

5287:

Cowling, V. F. A remark on bounded functions. Amer. Math. Monthly 66 (1959), 119–120.

Let $f(z)=\sum a_n z^n$ be regular in $|z|<1$. By a theorem of G. H. Hardy [Quart. J. Math. 44 (1913), 147–160], if $\sum |a_n|^2<\infty$, then as $|z|\to 1$

$$\sum |a_n z^n|=o((1-|z|)^{-\frac{1}{2}}).$$

Using Hardy's method of proof (essentially an application of the Schwarz inequality), the author shows that if $\sum n|a_n|^2<\infty$, then as $|z|\to 1$

$$\sum |a_n z^n|=o((-\log(1-|z|))^{\frac{1}{2}}).$$

M. S. Robertson (New Brunswick, N.J.)

5288:

Hoh, Chen-chih. The Szegö problem in the theory of schlicht functions. Sci. Record (N.S.) 2 (1958), 86–91.

By the Szegö problem [G. Szegö, Jber. Deutsch. Math. Verein. 31 (1922), 42–43] the author means the following. Let S_M be the class of holomorphic and univalent functions $W=f(z)$ such that $f(0)=0$, $f'(0)=1$, and $|f(z)|\leqq M$ in $|z|<1$. Let $W_k=\alpha_k(f)\exp[i((2k+1)/n+\varepsilon)\pi]$, $\varepsilon>0$, be the boundary point of the image under f which lies on the ray $\arg W=((2k+1)/n+\varepsilon)\pi$ and is nearest to $W=0$ ($k=1, 2, \cdots, n$). The problem is to determine $\min_f \max_{1\leqq k\leqq n} \alpha_k(f)$ $(0\leqq\varepsilon\leqq 2)$. In this paper he defines several subclasses of normalized univalent functions for which he can obtain precise results in the Szegö problem. The proofs of the theorems are not given, but the author states that they can be proved by the method of Grötzsch.

G. Springer (Lawrence, Kans.)

5289:

Mocanu, P. T. Sur un théorème de recouvrement dans la classe de fonctions univalentes. Gaz. Mat. Fiz. Ser. A (N.S.) 10(63) (1958), 473–477. (Romanian. French and Russian summaries)

Let S be the class of functions $w=f(z)$, holomorphic and univalent in $|z|<1$, satisfying $f(0)=0$ and $f'(0)=1$. The largest domain contained in the image of the domain D of the z-plane bounded by the circle $r=\cos\theta$ (in polar coordinates) is the convex domain of the w-plane whose boundary is given in polar coordinates (R_1, γ) by the parametric equations

$$R_1=\tfrac{1}{4}(1-t^4)t^{-2}\exp[2(1-t^6)(1+t^6)^{-1}\log|t|],$$
$$\gamma=2[\arctan t+2t^3(1+t^6)^{-1}\log|t|]$$

for $-1\leqq t\leqq 1$. *G. Springer* (Lawrence, Kans.)

5290:

Reade, Maxwell O. On Umezawa's criteria for univalence. II. J. Math. Soc. Japan 10 (1958), 255–259.

The author gives new proofs, with refinements, of criteria for univalence due to T. Umezawa [Tôhoku Math. J. (2) 7 (1955), 212–228; MR 17, 1068], V. S. Rogožin [Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 32 (1955), 135–137; MR 17, 724] and himself [Proc. Internat. Congr. Math. Amsterdam, 1954, p. 462, Noordhoff, Groningen, 1957; see also J. Math. Soc. Japan 9 (1957), 234–238; MR 19, 642].

M. S. Robertson (New Brunswick, N.J.)

5291:

Uluaçy, Cengiz. The exact values of the Bloch-Landau constants B, L. J. Reine Angew. Math. 199 (1958), 188–191.

Let R be the class of functions $w=f(z)$, regular for $|z|<1$, for which $|f'(0)|=1$. The sup of all numbers B for which the statement "$f\in R\Rightarrow f^{-1}(w)$ is regular on a w-disk of radius B" is true is called $\mathfrak{B}$. The sup of all numbers L for which the statement "$f\in R\Rightarrow f$ assumes all values in a w-disk of radius L" is true is called $\mathfrak{L}$. Upper bounds for $\mathfrak{B}$ were obtained by Ahlfors-Grunsky [Math. Z. 42 (1937), 671–673], and for $\mathfrak{L}$ by Rademacher [Amer. J. Math. 65 (1943), 387–390; MR 4, 270] and Robinson. These respective upper bounds had been conjectured to give the precise values of $\mathfrak{B}$ and $\mathfrak{L}$. The object of the present paper is to outline proofs of these conjectures. The proof for $\mathfrak{B}$ is only sketched. The proof for $\mathfrak{L}$ rests on a preliminary assertion (theorem 3) to the effect that the Riemann surface of an extremal function cannot contain branch points of finite order. {The proof of theorem 3, however, contains a serious gap: When the maximum principle is applied to $|f_z(z, \alpha)|_{z=0}$, the author neglects to consider the (non-trivial) possibility that $|f_z(0, \alpha)|=1$.} [Cf. also the abstracts by the author in Bull. Amer. Math. Soc. 54 (1948), 834–836.] *E. Reich* (Minneapolis, Minn.)

5292:

Scott, W. T. A covering theorem for univalent functions. Amer. Math. Monthly 64 (1957), no. 8, part II, 90–94.

Let U_+ denote the class of functions, including $f(z)\equiv z$, which are regular analytic and univalent in $|z|<1$ and normalized by $f(z)=z+a_2z^2+a_3z^3+\cdots$, in which the first non-zero coefficient a_m, $m\geqq 2$, satisfies $a_m>0$. For each $f\in U_+$, and for each ϕ, $0\leqq|\phi|\leqq\pi$, let $\rho_f(\phi)$ denote the distance, in the direction ϕ, from $w=0$ to the nearest boundary point of the image of $|z|<1$. Define $\rho(\phi)=\inf_{f\in U_+}\rho_f(\phi)$. The author then proves that $\rho(\phi)=\frac{1}{4}$ for $0\leqq|\phi|\leqq\pi/2$; $\frac{1}{4}|\sin\phi|<\rho(\phi)\leqq\frac{1}{4}$ for $\pi/2<|\phi|\leqq 3\pi/4$; $[4|\cos\phi|]^{-1}<\rho(\phi)\leqq\frac{1}{2}$ for $3\pi/4\leqq|\phi|<\pi$; and $\rho(\pi)=\frac{1}{4}$. For $0\leqq|\phi|<\pi/2$, $\rho(\phi)e^{i\phi}$ is not an omitted value for any function of U_+, but for $\pi/2\leqq|\phi|\leqq\pi$, each value $\rho(\phi)e^{i\phi}$ is an omitted value for some function of U_+.

G. Springer (Lawrence, Kans.)

5293:

Tammi, Olli. Note on Gutzmer's coefficient theorem. Rev. Fac. Sci. Univ. Istanbul Sér. A 22 (1957), 9–12. (Turkish summary)

The author proves from first principles that if $f(z)=\sum_0^\infty a_nz^n$ maps $|z|<R$ onto a domain G lying in a strip of width δ, then the sharp inequality

$$\sum_1^\infty |a_n|^2R^{2n}<\tfrac{1}{4}\delta^2$$

holds. He points out that this also follows from subordination. *W. K. Hayman* (London)

5294:

Vostrecov, B. A.; and Ignat'eva, A. V. On the degree of approximation to analytic functions on arbitrary continua. Moskov. Oblast. Pedagog. Inst. Uč. Zap. 57 (1957), 45–50. (Russian)

Let E be a bounded continuum with connected complement and capacity 1. Let $z=\phi(w)$ map $|w|>1$ on the exterior of E. Let L_h be the level lines (images of $|w|=1+h$), D_h the finite region bounded by L_h and ψ the inverse of ϕ. If $f(z)$ is represented in D_h by

$$f(z)=(2\pi i)^{-1}\int_{L_h}(t-z)^{-1}F(t)dt,$$

the authors use the partial sums of the expansion of a function in series of Faber polynomials to obtain an upper estimate for $\rho_n(f, \bar{D}_r)$, the best approximation to f by polynomials of degree n on $\bar{D}_r$ $(r<h)$. In particular,

$$\rho_n(f, E)<Cn^5(1+h)^{-n}\int_{L_h}|F|dt.$$

[Cf. Mergelyan, Dokl. Akad. Nauk SSSR **91** (1953), 1271–1274; Uspehi Mat. Nauk (N.S.) **7** (1952), no. 2(48), 31–122; Amer. Math. Soc. Transl. no. 101 (1954); MR **15**, 524; **14**, 547; **15**, 612.] *R. P. Boas, Jr.* (Evanston, Ill.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 5504.

5295:

✱Priwalow, I. I. Einführung in die Funktionentheorie. I. Mathematisch-Naturwissenschaftliche Bibliothek, 21. B. G. Teubner Verlagsgesellschaft, Leipzig, 1958. iv+163 pp. DM 7.30.

A German translation by Viktor Ziegler from the Russian of the 9th edition [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954], listed in MR **16**, 121. For a review of the 8th edition (1948) see MR **13**, 334.

5296:

Shimoda, Isae. Notes on the functions of two complex variables. J. Gakugei Tokushima Univ. **8** (1957), 1–3.

Let $f(x, y)$ be a complex-valued function defined for the complex variables $x\in D$ and $y\in D'$. Hartogs proved that if f is regular analytic in each variable separately, it represents an analytic function of two complex variables in the product domain (D, D'). The author studies now the case in which $f(x, y)$ is regular analytic in x for all fixed y and regular analytic in y for a sequence $x_n\in D$ which converges to a point $x_0\in D$. Using Vitali's theorem and elementary set theory, he shows that D' can be exhausted by a sequence of simply-connected domains E_n such that $f(x, y)$ is regular analytic in both complex variables in each product domain (D, E_n).

M. M. Schiffer (Stanford, Calif.)

5297:

Bochner, S. On identical vanishing of holomorphic functions in several complex variables. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 46–47.

This note is a generalization of a result of the reviewer. Let D be a bounded circular domain in C_n and let B_0 be an open subset of the boundary of D. Let $f_1, \cdots, f_m$ be holomorphic functions on D with continuous boundary values on B_0 and, finally, let $P_1, \cdots, P_m$ be homogeneous polynomials of the same degree in $\zeta_1, \cdots, \zeta_n$. The author proves the following.

Theorem 1: If

$$\Phi(\zeta; z)=\sum_{\mu=1}^{m}P_\mu(\zeta)f_\mu(z)$$

and if $\Phi(\bar{z}; z)=0$ on B_0, then Φ is identically 0. In particular, if $P_1, \cdots, P_m$ are linearly independent over the constants, then $f_1, \cdots, f_m$ are all identically 0.

Theorem 2: If D is the unit ball and if $P_1, \cdots, P_m$ are any polynomials in $\zeta_1, \cdots, \zeta_n$, then the vanishing of $\Phi(\bar{z}; z)$ on B_0 implies that

$$\Phi(\zeta; z)=(1-\zeta_1z_1-\zeta_2z_2-\cdots-\zeta_nz_n)\Psi(\zeta; z),$$

where $\Psi(\zeta; z)$ is a polynomial in (ζ_ν) whose coefficients are holomorphic functions on D.

J. J. Kohn (Waltham, Mass.)

5298:

Shutrick, H. B. Complex extensions. Quart. J. Math. Oxford Ser. (2) **9** (1958), 189–201.

The following theorem is proved: Every real analytic manifold V satisfying the second axiom of countability admits a complex extension X; i.e., roughly speaking, for each point $x\in V$ there exists a system of local complex coordinates z_i in X such that in a neighbourhood of x in X, V is the set of the points of X whose coordinates z_i are real, and, moreover, the real parts of the z_i constitute a coordinate system in V in some neighbourhood of x. The theorem is equivalent to the following statement (without proof) of C. Ehresmann [Proc. Internat. Congr. of Math. Cambridge, Mass, (1950), vol. 2, pp. 412–419, Amer. Math. Soc., Providence, R.I.; MR **13**, 574]: There exists a neighborhood N of the diagonal Δ in $V\times V$ which admits a complex analytic structure with Δ isomorphic to V as a real analytic submanifold. The theorem had also been conjectured by several other people, but, according to the author, "it may have been assumed that the proof is more obvious than in fact it is".

E. Vesentini (Princeton, N.J.)

5299:

Grauert, Hans. On Levi's problem and the imbedding of real-analytic manifolds. Ann. of Math. (2) **68** (1958), 460–472.

In 1911 E. E. Levi proved that the boundary of a domain of holomorphy must satisfy a condition called pseudoconvexity. Whether this condition is also sufficient for the domains of holomorphy was left open for more than forty years, until K. Oka gave a positive solution of this conjecture by Levi for an unbranched domain over C^n [Jap. J. Math., **23** (1953), 97–155; MR **17**, 82–83]. In the present paper the author discusses a generalization of Levi's conjecture for a strongly pseudo-convex subdomain G of an arbitrary complex analytic manifold, and succeeds in proving that G is holomorphically convex (Theorem 1). Further, if G admits a strongly plurisubharmonic function, then G is a Stein manifold (Theorem 2). The idea of his proof is rather simple, but the method depends deeply upon the technique of sheaves.

As an application of Theorem 2, the author discusses the imbedding of real-analytic manifolds. C. B. Morrey [#5504 below] first succeeded in showing that every compact real-analytic manifold is imbedded bi-real-analytically into a Euclidean space of sufficiently high dimension. The author generalizes this result to a para-compact manifold R and gives a simplified proof (Theorem 3). In the demonstration he used an unpublished result by R. Remmert on the imbedding of a Stein manifold into C^m with sufficiently large m, which is unnecessary if R is compact. *S. Hitotumatu* (Tokyo)

SPECIAL FUNCTIONS

See also 5315, 5558.

5300:

Kelisky, Richard P. On formulas involving both the Bernoulli and Fibonacci numbers. Scripta Math. **23** (1957), 27–35 (1958).

Put

$$F_n(x)=\tfrac{1}{2}\{(x+1)^n-(x-1)^n\},\quad L_n(x)=\tfrac{1}{2}\{(x+1)^n+(x-1)^n\}$$

define the sequences u_n, v_n as solutions of $y_{n+1}=y_n+y_{n-1}$ such that $u_0=0$, $u_1=1$, $v_0=2$, $v_1=1$. Also let B_n and E_n denote the Bernoulli and Euler numbers in the even suffix notation. The author proves first that

$$F_n(5^{-\frac{1}{2}}v_r u_r^{-1})=5^{-\frac{1}{2}(n-1)}u_{rn}u_r^{-n},$$
$$L_n(5^{-\frac{1}{2}}v_r u_r^{-1})=5^{-n/2}v_{rn}u_r^{-n},$$

where $r\geqq 1$, $n\geqq 0$ and the left members are symbolic. Next a formula involving B_n, u_n, v_n is obtained. Various special cases of this formula are noted; in particular, we cite

$$\sum_{2k\leq n}\binom{n}{2k}B_{2k}5^k u_r^{2k}u_{r(n-2k)}=\tfrac{1}{2}nu_r v_{r(n-1)}.$$

An analogous result containing E_{2k} is

$$\sum_{2k\leq n}\binom{n}{2k}E_{2k}5^k 4^{-k}v_{n-2k}=2^{1-n}.$$

It is noted that

$$u_{rn}=2(-i)^{rn}T_n(i^r v_r/2),$$
$$u_{r(n+1)}=(i)^{rn}u_r\sum_{k=0}^{n}P_k(i^r v_r/2)P_{n-k}(i^r v_r/2),$$

where $T_n(x)$ is the nth Chebyshev polynomial and $P_n(x)$ is the nth Legendre polynomial.

L. Carlitz (Durham, N.C.)

5301:

Gandhi, J. M. The coefficients of $\cosh x/\cos x$ **and a note on Carlitz's coefficients of** $\sinh x/\sin x$. Math. Mag. 31 (1957/58), 185–191.

The author derives a number of arithmetical properties of the coefficients S_{2n} and β_{2n} appearing in the series

$$\frac{\cosh x}{\cos x}=\sum_{n=0}^{\infty}S_{2n}\frac{x^{2n}}{(2n)!},\quad \frac{\sinh x}{\sin x}=\sum_{n=0}^{\infty}\beta_{2n}\frac{x^{2n}}{(2n)!}.$$

For example, he shows that S_{2n} is an even positive integer if $n\geqq 1$, and that S_{4n} is divisible by $4n-1$ if $4n-1$ is prime. The method involves equating coefficients in power series expansions.

A. L. Whiteman (Los Angeles, Calif.)

5302:

Ananda-Rau, K. On Hermite's doubly periodic functions of the third kind. J. Indian Math. Soc. (N.S.) 21 (1957), 67–72 (1958).

Hermite a introduit les fonctions doublement périodiques de troisième espèce. Ce sont des fonctions entières ou méromorphes satisfaisant aux égalités fonctionnelles

$$\varphi(u+2\omega)=e^{au+b}\varphi(u),\quad \varphi(u+2\omega')=e^{cu+d}\varphi(u)$$

(a, b, c, d étant des constantes). Ces fonctions généralisent les fonctions quasi-elliptiques (pour lesquelles a et c sont nulles). L'auteur, qui a déjà donné [cf. J. Indian Math. Soc. **19** (1955), 95–103 (1956); MR **18**, 29] une méthode de construction des fonctions quasi-elliptiques, donne ici une méthode généralisant la précédente et permettant d'obtenir ces fonctions comme sommes de séries de la forme

$$\sum_{-\infty}^{+\infty}q^{kn^2}e^{hn+2knix}\Phi(z+n\alpha);$$

où $q=e^{i\alpha}$, $\alpha=\beta+i\gamma$, $\gamma>0$, k entier $\geqq 1$, et $\Phi(z)$, méromorphe, étant telle que $\Phi(z+\pi)=c\Phi(z)$ et que $|\Phi(z)|<e^{A|Y|}$ si $|Y|>L$, $L>0$, $A>0$, $\gamma A<k$.

R. Campbell (Caen)

5303:

Rimskiĭ-Korsakov, B. S. A version of the construction of a theory of generalized gamma-functions based on the Laplace transform. Moskov. Oblast. Pedagog. Inst. Uč. Zap. 57 (1957), 121–141. (Russian)

The author starts from a function $f(p)$ such that $f'(p)/f(p)$ is the Laplace transform of a function $h(t)$ and uses it to construct a generalized Γ-function which possesses analogues of many familiar formulas; the ordinary Γ-function is the special case when $h(t)=1$ and $f(p)=p$. First suppose that $t|h(t)|<Me^{bt}$ with $b<1$; then the limit

$$C_f=\lim_{n\to\infty}\left\{\frac{f'(1)}{f(1)}+\frac{f'(2)}{f(2)}+\cdots+\frac{f'(n-1)}{f(n-1)}-\log f(n)\right\}$$

exists. It has the integral representations

$$C_f=\int_0^\infty\left(\frac{1}{e^t-1}-\frac{1}{te^t}\right)h(t)dt-\log f(1)$$
$$=\int_{1-0}^{\infty}\frac{f'(s)}{f(s)}d([s]-s)-\log f(1),$$

the second of which generalizes van der Pol's representation of Euler's constant (with a different proof) [Canadian J. Math. **6** (1954), 18–22; MR **15**, 525]. If $h(t)$ is bounded, as will be supposed from now on,

$$C_f=(2\pi i)^{-1}\int_{a-i\infty}^{a+i\infty}\{\log z-\psi(z)\}\frac{f'(1-z)}{f(1-z)}dz-\log f(1),\quad 0<a<1,$$

where ψ is the logarithmic derivative of Γ. The limit

$$\lim u_n=\lim_{n\to\infty}\left\{\prod_{k=1}^{n}\frac{f(2k)}{f(2k-1)}\right\}^2\Big/ f(2n+1)$$

exists and we have a generalization of Wallis's formula by putting

$$2\pi_f=(\lim u_n)\exp 2\int_0^\infty\{\tfrac{1}{2}(-1)^{[t]}-Sa(t)\}\frac{f'(t)}{f(t)}dt,$$

where $Sa(t)$ is the sawtooth function $[t]-t+\frac{1}{2}$.

The generalized Γ-function $\Gamma_f(p)$ is now defined by

$$\frac{\Gamma_f'(p)}{\Gamma_f(p)}=\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-pt}}{1-e^{-t}}\right)h(t)dt+\log f(1),\quad \Re(p)>0,$$

and $\Gamma_f(1)=1$; it satisfies $\Gamma_f(p+1)=f(p)\Gamma_f(p)$. The logarithmic derivative of Γ_f is ψ_f, and $\psi_f(1)=-C_f$. We have $\int_0^1\log\Gamma_f(s)ds=\frac{1}{2}\log(2\pi_f)$, generalizing Raabe's formula. There are a generalized multiplication formula,

$$\Gamma_{f(u/n)}(np)=\{2\pi_{f(u/n)}/(2\pi_{f(u)})^n\}^{\frac{1}{2}}\prod_{k=0}^{n-1}\Gamma_{f(u)}(p+k/n),$$

and a generalization of Knar's formula for $\Gamma(1+z)$ as a product of Γ-functions. The infinite product for $1/\Gamma(p)$ has the generalization

$$\frac{1}{\Gamma_f(p)}=f(p)\exp(C_f p)\prod_{n=1}^{\infty}\left\{\frac{f(n+p)}{f(n)}\exp(-pf'(n)/f(n))\right\}.$$

Five integral representations for

$$\log\Gamma_f(p)-\int_1^p\log f(z)dz+\tfrac{1}{2}\log f(p)+\int_0^\infty\frac{d}{dt}\frac{e^{-t}-1}{t}h(t)dt-\tfrac{1}{2}\log(2\pi_f)$$

generalize Binet's formulas for $\log\Gamma(p)$. If $h(t)$ is analytic at 0, Stirling's formula becomes the asymptotic ex-

pansion

$$\log \Gamma_f(p) \sim \int_1^p \log f(z)dz - \tfrac{1}{2}\log f(p) - \int_0^\infty \frac{d}{dt}\frac{e^{-t}-1}{t}h(t)dt + \tfrac{1}{2}\log(2\pi_f) + \sum_{n=0}^{\infty} n!c_n p^{-n-1},$$

where the c_n are combinations of Bernoulli numbers with the Maclaurin coefficients of h.

R. P. Boas, Jr. (Evanston, Ill.)

5304:

Levenson, M. E. A recursion formula for

$$\int_0^\infty e^{-t}(\log t)^{n+1}dt.$$

Amer. Math. Monthly **65** (1958), 695–696.

The recursion formula obtained for

$$I_n = \int_0^\infty e^{-t}(\log t)^n dt \quad (n=0, 1, 2, \cdots)$$

is

$$I_{n+1} = -\gamma I_n + \sum_{j=1}^{n} (-1)^{j+1} j! \binom{n}{j} \zeta(j+1) I_{n-j},$$

in which γ is Euler's constant and $\zeta(s)$ is the Riemann zeta function. This result is obtained in an elementary manner with the aid of some properties of the gamma function. *C. A. Swanson* (Vancouver, B.C.)

5305:

Klamkin, M. S. An application of the Gauss multiplication theorem. Amer. Math. Monthly **64** (1957), 661–663.

By interpreting the gamma functions in the Gauss-Legendre multiplication formula with n factors as Mellin transforms, the author evaluates integrals involving modified Bessel functions of the second kind: among others, the Laplace transform of $t^{-3/2}K_{1/3}[2t^{-\frac{1}{2}}]$, when $n=3$; and

$$\int_0^\infty K_{\frac{1}{2}}[2(x/u)^{\frac{1}{2}}]K_{\frac{1}{2}}[2u^{-\frac{1}{2}}]u^{-\frac{1}{2}}du,$$

when $n=4$.

G. Crane (Pittsburgh, Pa.)

5306:

Kuipers, L.; and Meulenbeld, B. On a generalisation of Legendre's associated differential equation. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 436–450.

This paper is concerned with the solutions of the differential equation

$$(1)\ (1-z^2)w'' - 2zw' + \left[k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)}\right]w = 0,$$

which, for $m=n$, reduces to Legendre's associated equation. The reviewer investigated the solutions of (1) in his thesis, but confined himself mainly to real z, $-1<z<1$. The authors' investigation concerns complex values of z and unrestricted k, m and n. They derive two solutions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ of (1), given in terms of contour integrals, which for $m=n$ reduce to Hobson's definitions of $P_k^m(z)$ and $Q_k^m(z)$. They also give representations in terms of hypergeometric functions.

D. J. Hofsommer (Amsterdam)

5307:

Al-Salam, W. A.; and Carlitz, L. Generalized Turán expressions for certain hypergeometric series. Portugal. Math. **16** (1957), 119–127.

Toscano ayant récemment démontré la formule suivante, relative aux polynomes d'Hermite

$$\sum_{0}^{2m} (-1)^r \binom{2m}{m-r} H_{n+r}(x)H_{n-r}(x) = \sum_{m}^{n} \binom{j-1}{m-1} \frac{H_{n-j}{}^2(x)}{(n-j)!},$$

les auteurs donnent deux autres preuves de cette formule et de beaucoup d'autres égalités analogues.

R. Campbell (Caen)

5308:

Carlitz, L. Note on orthogonal polynomials related to theta functions. Publ. Math. Debrecen **5** (1958), 222–228.

Continuing earlier work [Ann. Mat. Pura Appl. (4) **41** (1956), 359–373, MR **17**, 1205] the author gives new proofs of the orthogonal properties of the polynomials $H_n(x, q)$, $G_n(x, q)$; and proves also some related results many of which are again in the nature of orthogonal properties while others are integral representations of, or integral formulas for, the polynomials in question.

A. Erdélyi (Pasadena, Calif.)

5309:

Basoco, M. A. On certain arithmetical functions related to a non-linear partial differential equation. Enseignement Math. (2) **4** (1958), 32–40.

Van der Pol ayant étudié récemment [Nederl. Akad. Wetensch. Proc. Ser. A. **54**=Indagationes Math. **13** (1951), 261–271, 272–284; MR **13**, 135] la fonction

$$\alpha_{2k-1}(t) = \frac{\sum'_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (m+n\tau)^{-2k}}{\sum'_{m=-\infty}^{\infty} m^{-2k}}$$

où $-t=2\pi i\tau$, Im $\tau>0$, m, n entiers tels que $(m, n)\neq(0, 0)$; l'auteur considère ici les fonctions obtenues en supposant que m et n ne prennent plus toutes les valeurs entières, mais successivement que $(m, n)\equiv(0, 1)$ ou $(1, 0)$ ou $(1,1)$ mod 2.

On a ainsi 3 nouvelles fonctions $\psi_{2k-1}(t)$, $\chi_{2k-1}(t)$ et $\Phi_{2k-1}(t)$ dont l'auteur étudie les propriétés, d'abord celles qu'on obtient en se rappelant que la fonction $\theta(v, \tau)$ de Jacobi satisfait à l'équation aux dérivées partielles

$$\frac{\partial^2 z}{\partial s^2} = 2\frac{\partial z}{\partial t}\ (s=2\pi v,\ t=-2\pi i\tau)$$

puis les propriétés récurrentes. Il obtient les fonctions considérées comme sommes de séries doubles, et enfin montre que $\Phi_{2k-1}(t)$ est sa propre image (comme α_{2k-1}), dans la transformation modulaire (c'est-à-dire que $t^k\Phi_{2k-1}(2\pi t) = (-1)^k t^{-k}\Phi_{2k-1}(2\pi/t)$, $k>1$) alors que ψ_{2k-1} et χ_{2k-1} sont images réciproques l'une de l'autre. *R. Campbell* (Caen)

ORDINARY DIFFERENTIAL EQUATIONS

See also 5559, 5560, 5565.

5310:

Peyser, Gideon. On the Cauchy-Lipschitz theorem. Amer. Math. Monthly **65** (1958), 760–762.

The author presents a proof of the classical Cauchy-Lipschitz theorem concerning the existence and uniqueness of solutions of ordinary differential equations of the first order which differs in detail from the classical procedure in the use of an exponential contracting factor to prove convergence and an integral inequality to establish uniqueness. The same methods are used to obtain an estimate of the increment of the solution.

P. E. Guenther (Cleveland, Ohio)

5311:

Kooi, O. The method of successive approximations and a uniqueness-theorem of Krasnoselskiĭ and Kreĭn in the theory of differential equations. Nederl. Akad. Wetensch. Proc. Ser. A 61=Indag. Math. 20 (1958), 322–327.

Krasnoselskiĭ and Kreĭn proved the uniqueness of a solution of $y'=f(x, y)$ through the point (x_0, y_0) under the conditions: $f(x, y)$ continuous for $0\leqq x-x_0\leqq a$, $|y-y_0|\leqq b$; and

(1) $\quad (x-x_0)|f(x, y_1)-f(x, y_2)|\leqq k|y_1-y_2|;$

(2) $\quad |f(x, y_1)-f(x, y_2)|\leqq p|y_1-y_2|^\alpha;$

where $k>0$, $\alpha>0$, $0<k(1-\alpha)<1$ [Uspehi Mat. Nauk. (N.S.) 11 (1956), no. 1(67), 209–213; MR 18, 38]. The author proves the existence as well as the uniqueness under a set of general conditions which include (1) and (2) as a special case. *J. Elliott* (New York, N.Y.)

5312:

Kozlov, È. M. On reducing the order of a system of linear differential equations by its partial solution. Dopovidi Akad. Nauk Ukraïn. RSR 1958, 918–923. (Ukrainian. Russian and English summaries)

"The paper deals with two methods enabling one to find partial solutions of the n-vector system $\dot{x}=A(t)x$ when one of the characteristic roots is near zero. Least squares are used to obtain initial conditions such that numerical integration yields a slowly varying partial solution. Then the order of the system is lowered by the number of linearly independent partial solutions obtained." (From the author's summary).

S. Lefschetz (Mexico, D.F.)

5313:

Gröbner, Volfango. Nuovi contributi alla teoria dei sistemi di equazioni differenziali nel campo analitico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 23 (1957), 375–379.

By using the linear differential operator

$$(*)\quad D=\theta_1(z)\frac{\partial}{\partial z_1}+\theta_2(z)\frac{\partial}{\partial z_2}+\cdots+\theta_n(z)\frac{\partial}{\partial z_n},$$

where $\theta_i(z)$ are holomorphic functions of the complex variables $z_1, z_2, \cdots, z_n$, the author constructs the solutions of systems of first-order differential equations in the complex domain. If we have $dZ_i/dt=\theta_i(Z)$ $(i=1, 2, \cdots, n)$, $\theta_i(Z)$ holomorphic in a certain neighborhood of z, then $Z_i=\sum_{\nu=0}^{\infty}t^\nu/\nu!D^\nu z_i$. The operator (*) is also used to express the solutions of first-order partial differential equations in a series. *C. D. Calsoyas* (Livermore, Calif.)

5314:

Karasev, I. M. Linear differential equations with particular solutions of given type and subjected to given conditions. Kabardin. Gos. Ped. Inst. Uč. Zap. 12 (1957), 39–41. (Russian)

The following theorem is proved: If the particular solutions y_1 and y_2 of the equation (1) $y''+P(x)y'+Q(x)y=0$ are subjected to the condition $y_1y_2=C_1C_2n$ where C_1 and C_2 are arbitrary constants $\neq 0$ and n is a number $\neq 0$ then the necessary and sufficient condition that y_1 and y_2 have the form $y_1=C_1e^{D^p\xi}$, $y_2=C_2e^{D^p\eta}$ where ξ and η are functions of x, $D=d/dx$ and p is an integer, is that $P(x)=-DQ(x)/2Q(x)$. *M. Zlámal* (Brno)

5315:

Oğuztöreli, M. N. Sur les propriétés relatives à la distribution des valeurs qui correspondent aux solutions des équations différentielles linéaires à coefficients elliptiques. Rev. Fac. Sci. Univ. Istanbul Sér. A 22 (1957), 91–96. (Turkish summary)

The author studies the solutions of linear differential equations with doubly-periodic coefficients from the point of view of Nevanlinna's theory of the distribution of values. The properties of these solutions are shown to present many analogies to those of solutions of linear differential equations with rational coefficients.

Z. Nehari (Pittsburgh, Pa.)

5316:

Wintner, Aurel. A stability criterion for quasi-harmonic vibrations. Quart. Appl. Math. 16 (1958), 423–426.

Consider the equation (1) $x''+\omega^2(t)x=0$, where $\omega(t+\pi)=\omega(t)$ is a real continuous function of t. Using a result of G. Borg [Arkiv Mat. Astr. Fys. 31A (1944), no. 1; MR 8, 70], sufficient conditions, involving the deviation of $\omega^2(t)$ from a constant, are given for system (1) to be stable (i.e., all solutions are bounded for $-\infty<t<+\infty$). The results are naturally not as general as those of G. Borg, but may be somewhat easier to apply. In some cases, the results of this paper include the original result of A. Liapounoff namely, $\pi\int_0^\pi\omega^2(t)dt<4$. In applying the results to the Mathieu equation, $\omega^2(t)=a+b\cos 2t$, $a>|b|>0$, the results are incorrect and equation (20) should be $a^2<Q(a^2-b^2)^{\frac{1}{2}}$. *J. K. Hale* (Baltimore, Md.)

5317:

Bellman, Richard. On a generalization of a result of Wintner. Quart. Appl. Math. 16 (1958), 431–432.

It is proved that the general solution of the vector-matrix system $\dot{y}=B(t)y$ is approximated to leading order by the general solution $X(t)$ of $\dot{x}=A(t)x$ provided

$$\int^\infty\|B(t)-A(t)\|\,\|X(t)\|\,\|X^{-1}(t)\|dt<\infty.$$

T. M. Cherry (Melbourne)

5318:

Latyševa, K. Ya. Subnormal series as solutions of linear differential equations of arbitrary rank. Kiïv. Derž. Univ. Nauk Zap. 16 (1957), no. 2=Kiev. Gos. Univ. Mat. Sb. 9(1957), 119–135. (Russian)

5319:

Latyševa, K. Ya. Solutions in finite form of homogeneous linear differential equations with polynomial coefficients. Kiïv. Derž. Univ. Nauk Zap. 16 (1957), no. 2=Kiev. Gos. Univ. Mat. Sb. 9 (1957), 137–157. (Russian)

5320:

Kondrat'ev, V. A. The zeros of the solutions of equation $y^{(n)}+p(x)y=0$. Dokl. Akad. Nauk SSSR 120 (1958), 1180–1182. (Russian)

Continuing his study [same Dokl. 118 (1958), 22–24; MR 20 #146] of the equation (1) $y^{(n)}+p(x)y=0$ on $[a, b]$ or $[a, +\infty)$ with $p(x)$ continuous, the author now shows for general n: (I) If $p(x)\geqq q(x)>0$ and if every solution of $y^{(n)}+q(x)y=0$ either has infinitely many zeros or tends monotonically to zero, then the same alternative holds for every solution of (1). (II) If there exist functions $p_i(x)$, $i=1, 2$, such that $p_1(x)\leqq p(x)\leqq p_2(x)$ and the equations $y^{(n)}+p_i(x)y=0$ are non-oscillatory, then so is (1).

H. A. Antosiewicz (Los Angeles, Calif.)

5321:

Halanay, A.; and Shandor, Sh. Sturm type theorems for self-conjugate systems of higher order differential equations. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 506–507. (Russian)

The authors announce results generalizing the classical Sturm theory for the separation and monotonicity of conjugate points for the following situation. Let

$$L=\sum_{0\le i\le n}(-1)^i\frac{d^i}{dt^i}\left(\theta_{n-i}\frac{d^i}{dt^i}\right),\quad \theta_0>0,$$

be a system of ordinary differential operators of rank p, where each θ_j is a continuous symmetric matrix-valued function of t. Two points a and b on the t-axis are said to be associated if there exists a non-zero solution $y(t)$ of $Ly=0$ such that $d^jy(a)/dt^j=0$, $d^jy(b)/dt^j=0$ for $0\le j\le n-1$; conjugate if a and b are associated and if there exists no c with $a<c<b$ such that a and c are associated. A monotone sequence $\{t_j\}$ of points $(t_j<t_{j+1})$ is said to be a conjugate system if t_{j+1} is conjugate to t_j for all j $(-\infty<j<+\infty)$. The authors state the following. Theorem 1: If t_j and t_k' are two distinct conjugate systems for $Ly=0$, then for every j, there exists one and only one k such that $t_j<t_k'<t_{j+1}$. Theorem 2: Let $\bar L$ be another system of rank p with coefficients $\bar\theta_j$ such that $\theta_j\le\bar\theta_j$ for all j. Let $\{t_j\}$ and $\{\bar t_j\}$ be conjugate systems for L and $\bar L$, respectively, with $t_0=\bar t_0$. Then for all positive k, $\bar t_k\ge t_k$, $\bar t_{-k}\le t_{-k}$, and if the inequality $\theta_j<\bar\theta_j$ holds for at least one j, strict inequality holds between the t_k and $\bar t_k$ for every k. Theorem 3: Let

$$B=\sum_{i<m}(-1)^i\frac{d^i}{dt^i}\left(p_i\frac{d^i}{dt^i}\right)$$

be another system of rank p, with $p_j\ge 0$ for all j, $p_j>0$ for at least one j. Let $L_\lambda=L-\lambda B$, and $[a,b]$ be a fixed interval. Then there exists a sequence of real numbers $\{\lambda_k\}$ such that $\lambda_k\to\infty$ for which L_{λ_k} has a system of $(k+1)$ conjugate points on $[a,b]$ beginning at a and ending at b. Moreover, there exists a sequence $\{\lambda_k'\}$ such that the two point boundary value problem $(L-\lambda_k'B)y=0$ with $d^jy/dt^j=0$ at a and b, for $j<n$, has a non-zero solution, while b is a kth associate of a for $(L-\lambda_k'B)$.

These results are generalizations of previous results of Cimmino [Math. Z. **32** (1930), 4–58], Bliss and Schoenberg [Amer. J. Math. **53** (1931), 781–800], and R. L. Sternberg [Duke Math. J. **19** (1952), 311–322; MR **14**, 50]. The note concludes by stating some sufficient conditions for the non-existence of conjugate points on a half-axis.

F. Browder (New Haven, Conn.)

5322:

Conti, Roberto. Sulla t_∞-similitudine tra matrici e l'equivalenza asintotica dei sistemi differenziali lineari. Riv. Mat. Univ. Parma 8 (1957), 43–47.

Suppose $A(t)$, $B(t)$ are $n\times n$ matrices, L-integrable in every finite interval in $I=[t_0,+\infty)$ and let $T(t)$ be an $n\times n$ matrix absolutely continuous in every finite interval of I, bounded in I, with $T^{-1}(t)$ bounded in I. $A(t)$ and $B(t)$ are t^∞-similar if there exists a function $T(t)$ as above with

$$\int_{t_0}^{+\infty}\|T'(t)+T(t)\cdot A(t)-B(t)T(t)\|dt<+\infty,$$

where $'$ represents differentiation with respect to t. For applications of this concept to the stability of linear differential systems see R. Conti [Atti. Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 247–250; MR **18**, 483] and U. Barbuti [Boll. Un. Mat. Ital. (3) **12** (1957), 61–66; MR **19**, 416]. The systems (A) $y'=A(t)y$ and (B) $x'=B(t)x$ are asymptotically equivalent if to every solution $y(t)$ of (A) there exists an $x(t)$ satisfying (B) with $\lim_{t\to\infty}\|x(t)-y(t)\|=0$. (A) is uniformly stable if $\|Y(t)Y^{-1}(\tau)\|\le c$, $0\le\tau\le t$, c constant, $Y(t)$ is an $n\times n$ matrix solution of (A) with $Y(0)=E$, the identity matrix. (A) is reducible if $A(t)$ is t_∞-similar to a constant matrix.

Theorem 1: If (A) is (i) reducible and (ii) uniformly stable; (iii) $A(t)$ and $B(t)$ are t_∞-similar; (iv) $\lim_{t\to\infty}T(t)$ exists and $=T(\infty)$, $\det T(+\infty)\ne 0$; then (A) and (B) are asymptotically equivalent. This includes a result of A. Wintner [J. Math. Mech. **6** (1957), 301–309; MR **19**, 33] and an example is given to show that Theorem 1 is actually more general than Wintner's. Theorem 2. If the conditions of Theorem 1 are satisfied with (i) replaced by $\int_{t_0}^{+\infty}\|A'(t)\|dt<+\infty$, and p of the characteristic roots of $A(t)$ are distinct and have zero real parts in I and the remaining $n-p$ roots $\le -a^2<0$ in I, then (A) and (B) are asymptotically equivalent. *J. K. Hale* (Baltimore, Md.)

5323:

Bogdanov, Yu. S. Some tests for the absence of closed trajectories. Dokl. Akad. Nauk SSSR **120** (1958), 939–940. (Russian)

The system dealt with is

$$(1)\qquad \dot x_i=p_i(x_1,x_2)\quad (i=1,2),$$

where the p_i are of class C^2 in a region X, the set of critical points in X is X_D, $G\subset X$, $G'=X-X_D$, V is a bounded open region of X. One defines G as: (a) self-acyclic if G contains no closed trajectory D; (b) acyclic relative to V if no D surrounds V.

Let $h=\operatorname{diam} V$. Then (b) holds under the following conditions: (I) At every point of G' the curvature of the path through the point $>2/h$. (II) The paths D have no inflexion in G' and the set of segments from the points of G' to the centers of curvature of D does not cover V. (III) The set of normals to D from all points of G' does not cover V.

In addition (a) holds if: (IV) V does not meet X_D and the curvature of any path through any point of V is $2/h$; (V) No path through any point of G' has a point of stationary curvature in G'. *S. Lefschetz* (Mexico, D.F.)

5324:

Helms, L. L.; and Putnam, C. R. Stability in incompressible systems. J. Math. Mech. **7** (1958), 901–903.

Let $\dot x=f(x)$, where f is of class C^1 and $\operatorname{div} f=0$. The authors prove that every bounded, two-sided Lyapunov stable solution must be Bohr almost periodic. They give two simple proofs, one based on a result of Khintchine, the other on the mean ergodic theorem of von Neumann.

H. A. Antosiewicz (Los Angeles, Calif.)

5325:

Magiros, Demetrios G. Subharmonics of any order in nonlinear systems of one degree of freedom: application to subharmonics of order 1/3. Information and Control 1 (1958), 198–227.

This paper is an exposition of the classical small-parameter method in the spirit of Poincaré. The method is applied to the equation

$$\ddot Q+Q+\varepsilon(k\dot Q-c_1Q+c_2Q^2+c_3Q^3)=B\sin nt,$$

where solutions of period 2π are sought; the case $n=3$ is studied with greater detail. {In the opinion of the re-

viewer, the author's claim that the results apply for strong non-linearity seems unwarranted.}

J. L. Massera (Montevideo)

5326:

Villari, Gaetano. Sul carattere oscillatorio delle soluzioni delle equazioni differenziali lineari omogenee del terzo ordine. Boll. Un. Mat. Ital. (3) 13 (1958), 73–78.

Let (1) $y'''+p_1y''+p_2y'+p_3y=0$, where $p_i(x)$ are continuous functions for $x\geq\xi$, $p_2\leq 0\leq p_3$. Then if I is the class of all solutions such that for some $x\geq\xi$ either $y(x)y'(x)>0$ or $y'(x)y''(x)>0$, all the solutions in I are oscillatory or all non-oscillatory. If either $p_1\leq -1$, $p_3\leq -p_2$ or $|p_i|\leq K_ix^{-(2+i+\lambda_i)}$, $i=1, 2, 3$, $K_i, \lambda_i>0$, all the solutions of (1) are non-oscillatory. [Cf. M. Švec, Czechoslovak Math. J. **7**(82) (1957), 450–462; MR **20** #1816].

J. L. Massera (Montevideo)

5327:

Seifert, George. Acknowledgment: "On stability in the large for periodic solutions of differential systems." Ann. of Math. (2) **68** (1958), 473.

The author acknowledges the fact that theorem 1 of his paper under the given title [same Ann. (2) **67** (1958), 83–89; MR **19**, 960] is a special case of theorem 10 of a paper by D. C. Lewis [Amer. J. Math. **71** (1949), 294–312; MR **10**, 708; see also same J. **73** (1951), 48–58; MR **15**, 873].

R. G. Langenhop (Ames, Iowa)

5328:

Aizerman, M. A.; and Gantmacher, F. R. Determination of stability by linear approximation of a periodic solution of a system of differential equations with discontinuous right-hand sides. Quart. J. Mech. Appl. Math. 11 (1958), 385–398.

A translation into English of the Russian original in Prikl. Mat. Meh. **21** (1957), 658–669; MR **20** #153b.

5329:

Štelik, V. G. On the stability of solutions of systems close to the periodic. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 598–601. (Ukrainian. Russian and English summaries)

The author examines a system $dx/dt=A(\theta, \tau)x$, where $d\theta/dt=\nu(\tau)$, $\tau=\varepsilon t$, and the matrix $A(\theta, \tau)$ is real, periodic in θ with period ω_1, differentiable in θ and τ and bounded in τ for all $t\geq t_0\geq 0$, with coefficients satisfying certain conditions, making it possible to find a linear non-singular transformation reducing the initial system to a form which permits obtaining sufficient conditions for the stability and instability of the solutions of the initial system in Lyapunov's sense. *Author's summary*

5330:

Likova, O. B. On the behaviour of solutions of differential equations in the neighbourhood of closed orbits. Dopovidi Akad. Nauk Ukraïn. RSR **1957**, 535–538. (Ukrainian. Russian and English summaries)

"The author considers a system of differential equations

$$\frac{dx}{dt}=X(x)+\varepsilon X^*(t, x, \varepsilon),$$

where ε is a small parameter and x, X, X^* are vectors in Euclidean n-space. Making certain assumptions, the existence and uniqueness of an exact two-parametric family of particular solutions of the system is proved. This family of solutions has the property of attracting any solutions of the system whose initial values are sufficiently close to the above-mentioned two-parametric family of solutions." *Author's summary*

5331:

Zadiraka, K. V. On periodic solutions of a system of nonlinear differential equations with a small parameter by derivatives. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 131–134. (Ukrainian. Russian and English summaries)

"The author presents a demonstration of the existence of a unique periodic solution $x(t, \mu)$, $z(t, \mu)$ of the system

$$\frac{dx}{dt}=f(t, x, z);\ \mu\frac{dz}{dt}=F(t, x, z)$$

which, when $\mu\to 0$, tends uniformly in t to the solution $\bar{x}(t)$, $\bar{z}(t)$ of the degenerate system ($\mu=0$)

$$\frac{d\bar{x}}{dt}=f(t, \bar{x}, \bar{z});\ F(t, \bar{x}, \bar{z})=0."$$

Author's summary

5332:

Razumikhin, B. S. Stability in first approximation of systems with lag. J. Appl. Math. Mech. **22** (1958), 215–229 (155–166 Prikl. Mat. Meh.).

"Certain results are established in the theory o stability on the basis of first approximations for systems with lag, i.e.,

$$\frac{dx_i}{dt}=F_i(t; x_j(t); x_j(t-\tau))\quad (i=1, \cdots, n).$$

Sufficient conditions are obtained for first approximation stability of such systems." (Author's summary.)

T. M. Cherry (Melbourne)

5333:

Koronkevič, O. I. Structure of a particular solution of a system of linear differential equations with random free terms under resonance conditions. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 694–697. (Ukrainian. Russian and English summaries)

"The paper deals with the structure of the particular solution of a system or linear differential equations with random free terms in the case when the characteristic equation has zero or imaginary roots. Asymptotic formulae are given for the correlation matrix of the particular solution for the dispersion of a random function with stationary mth derivatives." *Author's summary*

5334:

Šolohovič, F. A. The relationship between a linear dynamical system and a certain differential equation in Banach space. Dokl. Akad. Nauk SSSR **120** (1958), 43–46. (Russian)

To every differential equation $dp/dt=Ap$, where A is a linear bounded operator in a Banach space, there corresponds a linear dynamical system $f(p, t)=f(t)p$ having the following properties: (A_1) $f(p, 0)=p$, i.e., $f(0)=I$; (A_2) $f[f(p, t_1), t_2]=f(p, t_1+t_2)$; ($A_3$) $f(p, t)$ depends continuously on its arguments; (A_4) $f(p_1+p_2, t)=f(p_1, t)+f(p_2, t)$. The author studies the systems defined axiomatically by (A_1)–(A_4) and is proving that in the finite-dimensional space the contrary is true, i.e., to every linear dynamical system corresponds a differential equation. This equivalence is not true in the infinite-dimensional space.

M. Zlámal (Brno)

5335:

Breus, K. A. On certain differential equations in a Banach space. Ukrain. Mat. Ž. **10** (1958), no. 2, 115–120. (Russian. English summary)

The results of a previous paper of the author [Dokl. Akad. Nauk SSSR **108** (1956), 997–1000; MR **18**, 212] which concerned the system of n linear differential

equations are now proved for the equation $dx/dt = A(\omega t)x$, where x is a vector in a Banach space and $A(\omega t)$ a linear operator with values in the same Banach space depending periodically on t with period $T=2\pi/\omega$, and ω is a large parameter. *M. Zlámal* (Brno)

PARTIAL DIFFERENTIAL EQUATIONS

See also 5313, 5427, 5504, 5609.

5336:

Bergman, Stefan. Properties of solutions of certain differential equations in three variables. J. Math. Mech. 7 (1958), 87–101.

The author considers the elliptic partial differential equation in three dimensions:

(1) $$\psi_{xx}+\psi_{yy}+\psi_{zz}+\tilde{F}(x, y, z)\psi=0.$$

Complex co-ordinates are introduced as $X=x$; $Z=\frac{1}{2}(y+iz)$; $Z^*=\frac{1}{2}(y-iz)$, so that equation (1) becomes

(2) $$\psi_{xx}+\psi_{zz^*}+F(X, Z, Z^*)=0.$$

Hypothesizing that F depends only on Z and Z^*, the author devises an integral operator $\phi_n(X, Z, Z^*; g)$ which maps each suitable analytic function $g(Z)$ into a solution of the partial differential equation (2). From ϕ_n one can construct a mapping τ_n which maps analytic functions $g(Z)$ into real solutions of equation (2). It is now possible to use known theorems in function theory to obtain results concerning solutions of the partial differential equation. The following quotation from the paper is worth noting.

"In the present and in ... preceding papers various theorems in the theory of analytic functions have been 'translated' into corresponding results for solutions of (2) and other linear partial differential equations. It is clear that, using integral operators, various other results of the theory of functions yield corresponding results in the theory of partial differential equations in two and three variables. In this connection it is not of interest to derive further examples of our procedure, but rather to consider various types of operators and analyze their usefulness for the study of partial differential equations. In doing so there arise the following problems. 1. The determination of various types of operators generating solutions of partial differential equations. 2. The classification of these operators and the investigation of properties of analytic functions which can be translated into theorems about partial differential equations by this or other types of operators. (It should be noted that often the same property of analytic functions yields completely different properties of the solutions when different operators are used.) 3. The study of operators generating pairs (or triples) of (real) solutions and the investigation of mappings of this type which can be considered to some extent as generalizations of conformal mappings."

R. B. Davis (Syracuse, N.Y.)

5337:

Kreyszig, Erwin. On coefficient problems of solutions of partial differential equations of the fourth order. J. Math. Mech. 6 (1957), 811–822.

The Bergman integral operator

$$V(z, z^*)=\int_{-1}^{1} E(z, z^*, t)f(z(1-t^2)/2)(1-t^2)^{-\frac{1}{2}}dt$$

maps analytic functions $f(z)$ one-to-one onto the set of solutions of the elliptic partial differential equation

(1) $$V_{zz^*}+AV_z+BV_{z^*}+CV=0.$$

As a result of this correspondence, one is able to study the behavior of V_{zz^*} if the coefficients V_{m0} in the expansion $V(z, z^*)=\sum_{m,n=0}^{\infty} V_{mn}Z^mZ^{*n}$ are known.

The analogous statements hold for the fourth order equation

(2) $$U_{zz^*zz^*}+aU_{zz}+bU_{zz^*}+cU_{z^*z^*}+dU_z+eU_{z^*}+hU=0,$$

using the Bergman integral operator

$$U(z, z^*)=\int_{-1}^{1} E_1(z, z^*, t)f(z(1-t^2)/2)(1-t^2)^{-\frac{1}{2}}dt + \int_{-1}^{1} E_2(z, z^*, t)g(z^*(1-t^2)/2)(1-t^2)^{-\frac{1}{2}}dt,$$

which maps pairs of analytic functions $f(z)$, $g(z)$ one-to-one onto the set of solutions of (2). In the expansion $U(z, z^*)=\sum_{m,n=0}^{\infty} U_{mn}Z^mZ^{*n}$, the behavior of $U(z, z^*)$ can be studied in relation to the special sequences $\{U_{m0}\}$ and $\{U_{m1}\}$.

The author studied $V(z, z^*)$ in relation to the sequence $\{V_{mn}\}$, for $n>0$, in a previous paper [J. Math. Mech. 6 (1957), 361–381; MR 19, 282]; in the present paper he studies the analogous problem for $U(z, z^*)$, that is, the fourth-order analogue of the previous second-order result.

R. B. Davis (Syracuse, N.Y.)

5338:

Kreyszig, Erwin. On some relations between partial and ordinary differential equations. Canad. J. Math. 10 (1958), 183–190.

The author considers the equation

(1) $$U_{zz^*}+B(z, z^*)U_{z^*}+C(z, z^*)U=0.$$

Solutions can be obtained as images of analytic functions under a mapping defined by a Bergman integral operator [cf. preceding review]. The images U_n of the analytic functions $f(z)=z^n$ are solutions of (1) with the special property that they are also solutions of a certain ordinary differential equation. Since the singularities of the coefficients of the ordinary differential equation can be related to the singularities of the coefficients of (1), it becomes possible to utilize known theorems about ordinary differential equations in studying the behavior of the U_n. *R. B. Davis* (Syracuse, N.Y.)

5339:

Kreyszig, Erwin. On singularities of solutions of partial differential equations in three variables. Arch. Rational Mech. Anal. 2 (1958), 151–159.

The results of the paper reviewed second above [#5337] are extended to second order elliptic equations in three space variables, of the form (in complex notation) $W_{xx}+W_{zz^*}+F(Z, Z^*)W=0$. (In real notation, this equation is of the form $V_{xx}+V_{yy}+V_{zz}+\alpha(y, z)V=0$.)

R. B. Davis (Syracuse, N.Y.)

5340:

Ahiezer, N. I. The uniqueness theorem for the heat equation. Har'kov. Politehn. Inst. Trudy. Ser. Inž.-Fiz. 5(1955), 51–55. (Russian)

5341:

Freud, Géza. Eine Eigenschaft der Lösungen parabolischer Differentialgleichungen. C. R. Acad. Bulgare Sci. 10 (1957), 451–452. (Russian summary)

Let $u(x, t)$ for $0<t\leqq T$, $0<x<l$ be a solution of the

parabolic differential equation

$$\partial^2 u/\partial x^2 + c(x, t)\partial u/\partial x = a(x, t)\partial u/\partial t,$$

where $a(x, t)$ is a positive function and $u(x, t)$ is continuous in the closed region $0 \leq t \leq T$, $0 \leq x \leq l$. The author establishes the following theorem:

Let $u(x, 0)$, $0 \leq x \leq l$ and $u(t, l)$, $0 \leq t \leq T$, be monotone non-decreasing and let $u(t, 0)$, $0 \leq t \leq T$ be monotone non-increasing. Then $u(x, t)$, for arbitrary t, $0 < t < T$, is a monotone non-decreasing function of x.

From the author's summary

5342:

Milicer-Grużewska, H. Un théorème limite sur la dérivée de l'intégrale de Poisson-Weierstrass généralisée. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 131–133.

This note concerns a property of the fundamental solutions of the parabolic partial differential equation

$$\partial u/\partial t = \sum_{i,j=1}^{n} a_{i,j}(x_1, \cdots, x_n, t)\partial^2 u/\partial x_i \partial x_j + \sum_{i=1}^{n} b_i(x_1, \cdots, x_n, t)\partial u/\partial x_i + c(x_1, \cdots, x_n, t)u,$$

where $(x_1, \cdots, x_n)$ belongs to a domain Ω of n-dimensional Euclidean space and $0 \leq t \leq T$. It is assumed that the quadratic form $\sum_{i,j=1}^{n} a_{i,j}x_i x_j$ is positive definite, and that the coefficients of the equation satisfy certain Hölder conditions. Denote the fundamental solution of the equation by $\Gamma(A, t, B, s)$, where A and B are two arbitrary points in Ω. The generalized Poisson-Weierstrass integral is

$$I(A, t, s) = \int\int\int_{\Omega} \Gamma(A, t, B, s) f(B, s) dB,$$

with f bounded and continuous in Ω. The author announces a formula for $\lim_{t\to s} \partial/\partial x_i \{I(x_1, \cdots, x_n, t, s)\}$. The proof is not given, but will be published later.

J. Elliott (New York, N.Y.)

5343:

Pul'kin, S. P. The Tricomi problem for the general equation of Lavrent'ev-Bicadze. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 38–41. (Russian)

The Tricomi problem for the equation $u_{xx} + \operatorname{sgn} y \cdot u_{yy} + A(x, y)u_x + B(x, y)u_y + C(x, y)u = 0$ ($C(x, y) \leq 0$ for $y > 0$) is studied. Under conditions that are too long to be stated here and that concern the boundary of the region, the coefficients of the equation and the given boundary values, the author proves the existence and uniqueness of the solution of this problem in a certain class of functions. In the special case $u_{xx} + \operatorname{sgn} y \cdot u_{yy} + C(x, y)u = 0$ ($C \leq 0$ for $y > 0$) the Tricomi problem always has a solution when $C(x, y)$ is dependent only on x or on y and the other conditions concerning the boundary and the boundary values are fulfilled.

M. Zlámal (Brno)

5344a:

Tanimura, Masayoshi. On the solution of some mixed boundary problems. IV. A solution of the problem of three sections of an infinite interval. Tech. Rep. Osaka Univ. **6** (1956), 265–271.

5344b:

Tanimura, Masayoshi. On the solution of some mixed boundary problems. V. The problems of alternate intervals. Tech. Rep. Osaka Univ. **7** (1957), 47–53.

5344c:

Tanimura, Masayoshi. On the solution of some mixed boundary problems. VI. Inside and outside problems of a circle on an infinite plate. Tech. Rep. Osaka Univ. **7** (1957), 55–63.

For parts I, II and III in the series, see same Rep. **5** (1955), 77–102, 337–348; **6** (1956), 63–74 [MR **17**, 629; **18**, 581]. For parts VII and VIII, see same Rep. **7** (1957), 297–306, 307–313 [MR **20** #2563a, *b*].

5345:

Sidorov, Yu. V. Solution of the Cauchy problem for the equation $\partial^2 u/\partial t^2 + \Delta\Delta u + 0$. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 341–348. (Russian)

The writer gives an explicit construction for the solution of the equation $\partial^2 u/\partial t^2 = \Delta\Delta u$ in the n-dimensional Euclidean space with the Cauchy boundary conditions $u|_{t=0} = 0$, $\partial u/\partial t|_{t=0} = \phi(x)$, by means of the n-dimensional Fourier transform.

F. Browder (New Haven, Conn.)

5346:

Babuška, Ivo. Eine Lösung des biharmonischen Problems im unendlichen Streifen. I. Numerische Ergebnisse und Anwendungen. Apl. Mat. **1** (1956), 34–43. (Czech. Russian and German summaries)

Verfasser betrachtet im offenen Streifen Q aller Punkte (x, y) mit $|y| < \pi/2$, das Anfangswertproblem der biharmonischen Differentialgleichung

$$\Delta\Delta U(x, y) = \frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0$$

unter den Bedingungen

$$\left|\frac{\partial U(x, y)}{\partial x}\right| < C(1+x^2)^{n/2}, \quad \left|\frac{\partial U(x, y)}{\partial y}\right| < C(1+x^2)^{n/2}, \; n < \infty \quad (C < \infty),$$

und gewinnt den Satz: sind $f'(x)$ und $g(x)$ stetige Funktionen, welche den Bedingungen

$$|f'(x)| < C(1+x^2)^{n/2}, \quad |g(x)| < C(1+x^2)^{n/2}$$

genügen, so existiert genau eine Lösung $U(x, y)$ des biharmonischen Anfangswertproblems im Sinne der angegebenen Definition; für $U(x, y)$ gilt auf der Streifengrenze $y = \pm\pi/2$

$$U\left(x, \pm\frac{\pi}{2}\right) = f(x); \quad \frac{\partial U}{\partial y}\left(x, \pm\frac{\pi}{2}\right) = \pm g(x).$$

$U(x, y)$ wird durch die uneigentlichen Integrale

$$U(x, y) = \int_{-\infty}^{+\infty} f(\xi)G_1(\xi - x, y)d\xi + \int_{-\infty}^{+\infty} g(\xi)G_2(\xi - x, y)d\xi;$$

$$-\infty < x < \infty; \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

dargestellt. Die uneigentlichen Integrale existieren im gewöhnlichen Sinne und die Greenschen Einflußfunktionen $G_i(x, y)$, $i = 1, 2$, ergeben sich gemäß

$$G_i(x, y) = \frac{\partial^2 H_i(x, y)}{\partial x^2}, \quad i = 1, 2,$$

wobei die Funktionen $H_i(x, y)$ Lösungen des biharmonischen Anfangswertproblems darstellen, welche den Bedingungen

$$H_1\left(x, \pm\frac{\pi}{2}\right) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}, \quad \frac{\partial H_1}{\partial y}\left(x, \pm\frac{\pi}{2}\right) = 0,$$

$$H_2\left(x, \pm\frac{\pi}{2}\right) = 0, \quad \frac{\partial H_2}{\partial y}\left(x, \pm\frac{\pi}{2}\right) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

genügen. Die Greenschen Einflußfelder G_t werden physikalisch gedeutet und numerisch ausgewertet.

M. Pinl (Cologne)

5347:

Zaharov, V. K. The first boundary problem for an elliptical type of equation of order four, degenerating at the domain boundary. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 694–697. (Russian)

Using methods developed by M. I. Višik [Mat. Sb. N.S. **35(77)** (1954), 513–568; MR **16**, 927] the writer gives detailed criteria for solvability for fourth order linear elliptic equations in two independent variables defined on a bounded domain, part of whose boundary is a piece of the x-axis, with coefficients and ellipticity degenerating on the x-axis.

F. Browder (New Haven, Conn.)

5348:

Browder, Félix. La théorie spectrale des opérateurs aux dérivées partielles du type elliptique. C. R. Acad. Sci. Paris **246** (1958), 526–528.

Soit A un opérateur différentiel linéaire d'ordre $2m$, uniformément elliptique dans R^n, à coefficients convenablement réguliers et bornés. Pour une fonction q assujettie à diverses conditions, l'A. donne des résultats précis sur la position du spectre et du spectre continu de l'opérateur minimal associé à $A+q$. L'A. annonce des évaluations dans les espaces L^p, $p\neq 2$, généralisation de résultats de Koševel [Mat. Sb. **38(80)** (1956), 359–372; MR **17**, 1213]. L'A. étend ensuite ses conclusions au cas où A est considéré sur un ouvert "régulier" de R^n. Pas de démonstrations.

J. L. Lions (Nancy)

5349:

Browder, Félix. Les opérateurs elliptiques et les problèmes mixtes. C. R. Acad. Sci. Paris **246** (1958), 1363–1365.

Etude du noyau de Green de l'opérateur A (notations du résumé précédent) pour le problème de Dirichlet. Etude du problème mixte avec conditions aux limites de Dirichlet.

J. L. Lions (Nancy)

5350:

Pederson, R. N. On the unique continuation theorem for certain second and fourth order elliptic equations. Comm. Pure Appl. Math. **11** (1958), 67–80.

L'A. établit un théorème d'unicité pour les équations semi-linéaires du 4ième ordre, de la forme

$$\Delta^2 u = f(x, u, u_{x^i}, u_{x^ix^j}, u_{x^ix^jx^k}), \tag{1}$$

où la fonction f est supposée satisfaire à des conditions du type de Lipschitz.

Désignant par m_{k+} la classe des fonctions $u\in C^k$ dont les dérivées d'ordre i, D^iu, satisfont à

$$\lim_{r\to 0} \exp(\alpha r^{-m})D^iu=0 \qquad (i=1, 2, \cdots, k-1),$$
$$\lim \exp(\alpha r^{-m})D^k u\in L^2,$$

quel que soit $\alpha>0$, il démontre que deux solutions u, v de (1), satisfaisant à $u-v\in m_{k+}$, sont identiques dans un voisinage de l'origine. La démonstration est basée sur des inégalités vérifiées par les fonctions de classe m_{2+} et m_{4+}.

J. Lelong (Paris)

5351:

Višik, M. I.; and Sobolev, S. L. General formulation of certain boundary problems for elliptic partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 521–523. (Russian)

The main concern of this note is of a conceptual nature, in the formulation of a very general class of problems rather than their solution. Its basic concept falls within a line of ideas originated in the 1930's by Leray, Sobolev, and Friedrichs and, somewhat later, exploited systematically in one of its aspects by the theory of distributions of L. Schwartz. The nucleus of this group of ideas is the definition of differential operators and boundary value problems by the use of the duality of linear spaces.

To put the matter in a more explicit and more general form than do the writers of the note, consider the following situation. Let E and F be Banach spaces, L a closed linear operator with domain D_L dense in E and range R_L contained in F. Let E^* and F^* be the conjugate spaces on E and F, respectively, i.e., the spaces of continuous linear functionals on E and F. If D_{L^*} consists of all functionals f^* from F^* for which $f^*(Le)$ is a bounded functional in the norm topology on D_L as a subset of E, then L^* may be defined uniquely on D_{L^*} by letting $L^*f^*=e^*$, where e^* is the unique continuous linear functional on E for which $e^*(e)=f^*(Le)$ for all e in D_L.

It is an easily established fact that if L maps D_L in a one-one fashion onto all of F, then L^* maps D_{L^*} in a one-one fashion onto all of E^*. Indeed if $L^*f^*=0$ for some f^* in D_{L^*}, then $f^*(Le)=(L^*f^*)(e)=0$ for all e in D_L, and since $R_L=F$, f^* must be zero. On the other hand, by the closed graph theorem, L^{-1} is a bounded linear mapping of F into E. Thus, given e^* in E^*, we may define a bounded linear functional f^* on F by $f^*(f)=e^*(L^{-1}f)$. Since for the f^* so defined, we have $f^*(Le)=e^*(L^{-1}Le)=e^*(e)$ for all e in D_L, it follows that $L^*f^*=e^*$, and L^* is both one-one and onto.

If it is the case that for a subset F_1 of F^* we already have a mapping L_1 of F_1 into E^* for which $(L_1f_1)(e)=f_1(Le)$ for all e in D_L, then L^* is an extension of L_1, the extension of L_1 with respect to duality with L.

In their note the writers carry through the construction of an extension by duality in several cases involving second-order elliptic operators of the form $\Delta+c(x)$, with Δ the Laplace differential operator. The first of these examples is typical: E is the subset of functions $e(x)$ from $C^m(\Omega)$, for a reasonably smooth bounded domain in Euclidean space, for which the boundary condition $\partial e/\partial n+b(x)e=0$ is satisfied on the boundary Γ of Ω. The space F is $C^k(\Omega)$, and one must assume that $m\leq m(k)$, $b(s)\geq c>0$ on Γ. The operator L is the Laplace operator, and its domain D_L consists of all e in E for which Le lies in $C^k(\Omega)$. Under the hypotheses, L maps D_L in a one-one fashion onto F, and by the preceding remarks, L^* maps D_L^* in a one-one fashion onto E^*. But F^* is the space of distributions of order k on the closure of Ω, while to every distribution of order m on the closure of Ω, there corresponds uniquely an element of E^*. In particular, if $L^*u=f-\phi(s)\delta_S$, where δ_S is the unit distribution of mass according to surface area on the boundary Γ, then u may be considered as the generalized solution of the boundary value problem

$$\begin{cases}\Delta u=f(x) \text{ in } \Omega,\\ \partial u/\partial n+b(s)u=\phi \text{ on } \Gamma.\end{cases}$$

The justification of this nomenclature is that every ordinary solution of the latter boundary value problem satisfies the condition

$$(u, Lv)=\int_\Omega u\Delta v dx=(p, v),\ v \text{ in } E,$$

where p is the distribution $f-\phi\delta_S$.

F. Browder (New Haven, Conn.)

5352:

Lions, Jacques-Louis. Conditions aux limites de Visik-Soboleff et problèmes mixtes. C. R. Acad. Sci. Paris 244 (1957), 1126–1128.

Riprendendo un'idea rapidamente enunciata da M. I. Vishik e S. L. Sobolev [#5351] in un caso particolare, l'A. svolge una teoria assai generale dei problemi al contorno per equazioni ellittiche e dei problemi misti secondo Hadamard, in classi di soluzioni a "integrale di Dirichlet" non finito e con condizioni al contorno non omogenee.

Sia Ω un aperto di R^n e V uno spazio di Hilbert tale che $H_0{}^m(\Omega)\subset V\subset H^m(\Omega)$ ($H^m(\Omega)$ spazio delle funzioni $u\in L^2(\Omega)$ aventi derivate, nel senso delle distribuzioni, d'ordine $\leqq m$ appartenenti a $L^2(\Omega)$; $H_0{}^m(\Omega)$ chiusura in $H^m(\Omega)$ dello spazio delle funzioni indefinitamente differenziabili a supporto compatto in Ω). Se Ω è sufficientemente "regolare" è possibile porre un isomorfismo topologico tra il duale di $H^k(\Omega)$ (k intero $\geqq 0$) e il sottospazio $H_{\bar\Omega}{}^{-k}$ delle distribuzioni di $H^{-k}(R^n)$ (duale di $H^k(R^n)$) aventi supporto in $\bar\Omega$. Siano allora $a(u,v)$ una forma sesquilineare e continua su $V\times V$, $a^*(u,v)=\overline{a(v,u)}$, Au l'operatore definito da $a(u,v)$, A^*u il suo aggiunto e si supponga che

$$A=\sum(-1)^pD^p(a_{pq}(x)D^q)\quad(|p|,\ |q|\leqq m)$$

con $a_{pq}(x)$ restrizione a Ω di $\mathscr{A}_{pq}(x)$ funzione indefinitamente differenziabile in R^n e limitata con tutte le sue derivate. Siano ancora N e N^* gli spazi delle $u\in V$ tali che sia rispettivamente $Au\in L^2(\Omega)$, $a(u,v)=\int_\Omega Auv\,dx$ per ogni $v\in V$, $A^*u\in L^2(\Omega)$, $a^*(u,v)=\int_\Omega A^*uv\,dx$ per ogni $v\in V$. Si ha allora il seguente Teor. I: Nelle ipotesi: 1) Ω sufficientemen te "regolare" (onde si possa stabilire l'isomorfismo di cui sopra);

2) $$\operatorname{Re} a(u,u)+\lambda\int_\Omega u^2dx\geqq\alpha\|u\|_V{}^2\quad(\alpha>0)$$

per ogni $u\in V$ e per un λ reale fissato (di qui segue [v. Lions, Acta Math. **94** (1955), 13–153; MR **17**, 745] che $A^*u+\lambda u$ è un isomorfismo di N^* su $L^2(\Omega)$, di cui indicheremo con G^* l'isomorfismo inverso); 3) per ogni $f\in H^k(\Omega)$, $G^*f\in H^{k+2m}(\Omega)\cap N^*$; allora, posto $\mathscr{A}=\sum(-1)^pD^p(\mathscr{A}_{pq}(x)D^q)$, per ogni $T\in H_{\bar\Omega}{}^{-(k+2m)}$ esiste una e una sola $U\in H_{\bar\Omega}{}^{-k}$ tale che $\mathscr{A}U-T\in M^k$, dove M^k è il sottospazio di $H_{\bar\Omega}{}^{-(k+2m)}$ delle distribuzioni S per cui $\langle S,\bar v\rangle=0$ per ogni v di $H^{k+2m}(R^n)$, la cui restrizione a Ω sia in N^*. L'ipotesi 3) è verificata in particolare, in virtù di risultati di F. E. Browder [Comm. Pure Appl. Math. **9** (1956), 351–361; MR **19**, 862], L. Nirenberg [Comm. Pure Appl. Math. **8** (1955), 649–675; MR **17**, 742], G. Stampacchia [Ricerche Mat. **5** (1956), 3–24; MR **18**, 579], per una vasta classe di problemi "uniformi" (Dirichlet, Neumann, $\cdots$) per i quali vale dunque il Teor. I.

Utilizzando il Teor. I si possono studiare anche i problemi misti secondo Hadamard; indicando con $\mathscr{D}_+'(t,E)$ lo spazio delle distribuzioni in t (variabile temporale) a valori in uno spazio E e a supporto limitato a sinistra, si ha il Teor. II: Nelle ipotesi 1), 2), 3), per ogni $T\in\mathscr{D}_+'(t, H_{\bar\Omega}{}^{-(k+2m)})$ esiste una e una sola $U\in\mathscr{D}_+'(t,H_{\bar\Omega}{}^{-k})$ tale che $(\mathscr{A}+\partial/\partial t)U-T\in\mathscr{D}_+'(t.M^k)$. Se $a(u,v)=\overline{a(v,u)}$ teoremi analoghi valgono anche per operatori del tipo $\mathscr{A}+\partial^2/\partial t^2+r(t)\partial/\partial t+s(t)$, r e s essendo funzioni indefinitamente differenziabili di t.

In tutti questi risultati la soluzione U è indefinitamente differenziabile se lo è la T e se la 3) vale per ogni k intero positivo.

E. Magenes (Genova)

5353:

Ladyženskaya, O. A. On integral estimates, convergence, approximate methods, and solution in functionals for elliptic operators. Vestnik Leningrad. Univ. 13 (1958), no. 7, 60–69. (Russian. English summary)

Let Ω be a region in n-space, and u a function vanishing on the boundary of Ω. $W_2{}^m(\Omega)$ is the totality of all u for which

$$\|u\|^2{}_{W_2{}^m\Omega}=\int_\Omega\sum_{k=0}^m\Big(\sum_{i_1+i_2+\cdots+i_n=k}\frac{\partial^ku}{\partial x_1{}^{i_1}\partial x_2{}^{i_2}\cdots\partial x_n{}^{i_n}}\Big)^2dx$$

is defined and finite. Let L be the elliptic operator

$$L(u)=\sum_{i,j}a_{ij}\frac{\partial^2u}{\partial x_i\partial x_j}+\sum_i a_i\frac{\partial u}{\partial x_i}+au.$$

Refining the author's previous work ["The mixed problem. for a hyperbolic equation", Gosudarstv. Izdat. Tehm-Teor. Lit., Moscow-Leningrad, 1953; MR **17**, 160], the following estimate is offered:

$$\|u\|_{W_2{}^m(\Omega)}{}^2\leqq c[\|Lu\|_{W_2{}^{m-2}(\Omega)}{}^2+\|u\|_{L_2(\Omega)}{}^2].$$

The constant c depends not on Ω, but on the maximal moduli of the derivative of the a_{ij} of order $\max(1, m-2)$, and of the derivatives of the a_i and a of order $m-2$; on α, the constant of ellipticity of L, and on the maximal moduli of the derivatives of order 1 to m of the function $y_n=w(y_1,\cdots,y_{n-1})$ which describes the boundary S of Ω in local coordinates.

The same type of estimate obtains for functions u satisfying a boundary condition of the form $\partial u/\partial l+\delta u|_S=0$, where $\partial u/\partial l$ is a directional, non tangential derivative, ("leaving" Ω) and δ is a bounded linear operator in $L_2(S)$.

The bridge from earlier work to the present is built of approximations using functions for which the earlier results hold.

The principal application lies in the proof that least squares, Ritz and Galerkin methods for solving boundary value problems yield approximants which converge relative to $\|\cdots\|_{W_2{}^m(\Omega)}$. *B. R. Gelbaum* (Minneapolis, Minn.)

5354:

Lions, Jacques-Louis. Equations d'Euler-Poisson-Darboux généralisées. C. R. Acad. Sci. Paris 246 (1958), 208–210.

The author studies mixed problems (in cylindrical domains) for equations of the form:

(1) $$A(t)\mathbf{u}(t)+\frac{\partial^2}{\partial t^2}\mathbf{u}(t)+\frac{c}{t}\frac{\partial}{\partial t}\mathbf{u}(t)+M(t)\frac{\partial}{\partial t}\mathbf{u}(t)=\mathbf{f}(t),$$

where $A(t)$ and $M(t)$ are linear operators (depending on t) from a Hilbert space V into a Hilbert space H (c is any complex number). Using the space $\mathscr{D}_0(E)$ of functions $\varphi(t)$, C^∞ in t, with values in the Hilbert space E, such that $\varphi^{(m)}(0)=0$ for all m, the main theorem states that, under suitable conditions on $A(t)$ and $M(t)$, for any $\mathbf{f}(t)\in\mathscr{D}_0(H)$ there is a unique $\mathbf{u}(t)\in\mathscr{D}_0(V)$ satisfying (1).

The method is partially based on transmutation operators, which transform $\partial^2/\partial t^2+\partial/\partial t$ into $\partial^2/\partial t^2$. Using such operators, one derives the solution of the above mixed problem from the one of a corresponding problem for

$$A(t)\mathbf{u}(t)+\frac{\partial^2}{\partial t^2}\mathbf{u}(t)+M(t)\frac{\partial}{\partial t}\mathbf{u}(t)=\mathbf{f}(t).$$

The theory can be applied in particular to partial

differential operators

$$A\left(x, t; \frac{\partial}{\partial x}\right)+\frac{\partial^2}{\partial t^2}+\frac{c}{t}\frac{\partial}{\partial t}+M(x, t)\frac{\partial}{\partial t},$$

where $A(x, t; \partial/\partial x)$ is a self-adjoint elliptic operator of order $2m$, V being a space of functions with finite Dirichlet integral. The solution obtained is then a solution in the classical sense.

Sketches of proofs are given. The methods rely mainly on previous results of the author.

J. F. Treves (Berkeley, Calif.)

5355:

Lopatins'kiĭ, Ya. B. Uniqueness of the solution of Cauchy's problem for a class of elliptic equations. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 689–693. (Ukrainian. Russian and English summaries)

"A system of equations (in matricial form)

$$(1) \qquad A\left(\frac{\partial}{\partial x}\right)u-F(x)=0$$

is considered in a bounded domain D of a real space of points $(x_1, \cdots, x_{s+1})$, the boundary of which consists of a surface T and a portion of the plane $x_{s+1}=h$.

In formula (1),

$$A(\alpha)=\sum_{k_1+\cdots+k_{s+1}=s} A_{k_1\cdots k_{s+1}}\alpha_1^{k_1}\cdots\alpha_{s+1}^{k_{s+1}},$$

$A_{k_1\cdots k_{s-1}}$ are $p\times p$ matrices with complex elements, $F(x)$ is a $p\times p$ matrix with complex-valued functions continuous in $\bar{D}$ as elements.

The following theorem is proved. If $\det A(\alpha)\neq 0$ for real $\alpha\neq 0$ and the system of equations $\partial \det A(\alpha)/\partial\alpha_k=0$ $(k=1, \cdots, s+1)$ has a zero-solution only, and if, further, $u(x)$ is a solution of the above equations with s-derivatives continuous in $\bar{D}$, the derivatives of which to the order $s-1$ are equal to zero on T, then $u(x)=0$ in D."

Author's summary

5356:

Vološina, M. S. On some properties of one class of strongly elliptical systems. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 913–917. (Ukrainian. Russian and English summaries)

A self-adjoint system of Euler equations of a variational problem is considered. The author establishes a connection between the fundamental matrix of the system and the kernel of a potential integral for a strongly elliptic system. The connection permits the application of the Neumann-Kellog method of solving the Dirichlet problem to such systems.

H. P. Thielman (Ames, Iowa)

5357:

Vološina, M. S. Properties of a class of strongly elliptical systems of partial differential equations with variable coefficients. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 1033–1037. (Ukrainian. Russian and English summaries)

"In this paper the theorem of the author [#5356 above] on the connection between the fundamental matrix and the nucleus of the integral of potential type is applied to the case of a self-adjoint system of Euler equations corresponding to the fundamental variational problem for a positive definite functional."

From the author's summary

5358:

Szmydt, Z. Sur l'existence de solutions de certains problèmes aux limites relatifs à un système d'équations différentielles hyperboliques. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 31–36.

Let $y=\gamma(x)$ and $x=\lambda(y)$ be two continuous curves in a rectangle D of the plane. Let U, P, Q denote n-vectors and suppose that $F(x, y, U, P, Q)$, $G(x, U, Q)$, and $H(y, U, P)$ are continuous functions for (x, y) in D and U, P, Q arbitrary. It is desired to find a solution for the system, denoted by

$$U_{xy}=F(x, y, U, U_x, U_y),$$

and satisfying the conditions

$$U(x_0, y_0)=U_0, \text{ a constant vector;}$$
$$U_x=G(x, U, U_y) \text{ on } y=\gamma(x);$$
$$U_y=H(y, U, U_x) \text{ on } x=\lambda(y).$$

It is assumed that F, G, and H do not grow faster than a linear function in U, P, Q and satisfy Lipschitz conditions in P, Q with suitably small coefficients. The proof uses J. Schauder's theorem on fixed points [Studia Math. 2 (1930), 171–180].

R. G. Bartle (Urbana, Ill.)

5359:

Prudnikov, A. P. The solution of a mixed boundary problem in the thermodiffusion theory. Dokl. Akad. Nauk SSSR **119** (1958), 249–251. (Russian)

This paper gives a method of solution for a pair of equations of the Fourier type. The solution is represented as a sum of integrals by the joint application of the Fourier and Laplace transforms. These integrals contain unknown functions which are determined from the boundary conditions through a system of Volterra integral equations.

C. G. Maple (Ames, Iowa)

5360:

Asadullin, È. A. Existence of analytic integrals of partial differential equations of non-normal type. Kazansk. Gos. Ped. Inst. Uč. Zap. **10** (1955), 169–179. (Russian)

5361:

Langenbah, A. On the application of a variational principle to some non-linear differential equations. Dokl. Akad. Nauk SSSR **121** (1958), 214–217. (Russian)

The author deals with equations

$$(*) \qquad Pu=f,$$

where $f\in M$ is a given element of a linear space M and P is a given nonlinear operator defined for every element $u\in M$. The solutions of (*) are sought in a given subspace $M_0\subset M$ which is dense in M. The usually defined derivative $P'(x)y$, $x, y\in M$, is a linear operator in y. The following assumptions are made: $P(0)=0$; $(P'(x)h_1, h_2)=(P'(x)h_2, h_1)$ for $x\in M$, $h_1, h_2\in M_0$; $(P'(x)h, h)>0$ for $x\in M$, $h\in M_0$, $h\neq 0$. Under these assumptions, if (*) has a solution, then this solution is unique and gives a minimum value to the functional

$$\Phi(u)=\int_0^1 (P'(t)u, u)dt-(f, u).$$

Also, $\Phi(u)$ is convex. Under the stronger assumption

$$(P'(x)h, h)\geq\gamma^2\|h\|^2,$$

it is also proved that every minimizing sequence for Φ has a limit, not necessarily in M_0, which is called a generalized solution for (*). If $\Phi(u)=F(u)+lu$, where l is a linear operator, $F(u)\geq\gamma^2\|u\|^2$, $F(2u)\leq kF(u)$, γ, k constants, $F(0)=0$, $F(-u)=F(u)$, $F(u)+F(v)-2F((u+v)/2)\geq 2F((u-v)/2)$, and $F(u-v)\to 0$ implies $|F(u)-F(v)|\to 0$, then $\Phi(u)$ has a minimum. Two examples of second order partial differential nonlinear equations are considered where the previous considerations apply. These equations

concern the mechanics of plastic solids, in particular the plastic torsion of rods, and had been studied by L. M. Kačanov. [Pertinent reference: L. M. Kačanov, Prikl. Mat. Meh. 12 (1948), 375–384; 13 (1949), 381–390; MR 10, 170; 11, 284.] *L. Cesari* (Baltimore, Md.)

5362:

Trèves, François. Domination et problèmes aux limites de type mixte. C. R. Acad. Sci. Paris **245** (1957), 2454–2457.

Let t be a real variable; $D^r=\partial^r/\partial t^r$ for $r\geq 0$ and $D^r=Y^{*(-r)}$ for $r<0$, where Y is the Heaviside function. Let E be a Hilbert space; call $D(E)$ the space of C^∞ maps $t\to E$ with the Schwartz topology. Let $p(t)$ be a C^∞ function such that $|p'(t)|\geq 1$ for all t. For any k we define the inner product

$$(\varphi,\psi)_{E;p,k}=\int(\exp(-p(t))D^k\varphi,\exp(-p(t))D^k\psi)_E dt$$

for $\varphi,\psi\in D(E)$. The completion of $D(E)$ for this norm is denoted by $D^k(p;E)$. The dual of $D^k(p;E)$ is isomorphic to $D^{-k}(p;E')$. Denote by $L(t,D)$ the integral-differential operator $\sum_{r=-\mu}^m B_r(t)D^r$, where $m\geq 1$, $\mu\geq 0$ and $B_r(t)$ is a C^∞ map $t\to$space of continuous maps of E into E. Let $V\subset E$ be a Hilbert space such that the injection $V\to E$ is continuous and V is dense in E. Finally, let $a(t;u,v)$ be bilinear for $(u,v)\in D'(V)\times V$.

The main result of the author is that under certain restrictions on a we can find a p_k such that given any $g\in D^k(p;E)$ there corresponds a unique $f\in D^k(p_k;V)$ which is a solution of the equation

$$(L(t,D)f,v)_E+a(t;f,v)=(g,v)_E$$

for all $v\in V$. *L. Ehrenpreis* (Waltham, Mass.)

5363:

Trèves, François. Domination et opérateurs hyperboliques. C. R. Acad. Sci. Paris **246** (1958), 680–683.

Notations: x, y are variables on Euclidean n-space; $y^0=(y_2,\cdots,y_n)$. $P(x,y)$ is a polynomial in y; $P(x,D)$ is obtained from $P(x,y)$ by replacing y_j by $(2\pi i)^{-1}(\partial/\partial x_j)$; $P(x_0,D)$ is the constant coefficient operator with coefficients evaluated at $x=x_0$. $P(D)$ is called normal (in y_1) if P is of degree m, $P(y)=a_0y_1{}^m+Q(y)$ with $a_0\neq 0$ and $Q(y)$ of degree $<m$ in y_1. P_m is the homogeneous part of degree m.

$P(x,D)$ is said to equidominate the family $\{Q_i(x,D)\}$ on an open set Ω in R^n if for each $\varepsilon>0$ we can find an $h\in R^n$ so that $\|\exp(-x\cdot h)Q_i(x,D)\varphi\|_{L^2}\leq\varepsilon\|\exp(-x\cdot h)P(x,D)\varphi\|_{L^2}$ for all i, φ indefinitely differentiable of compact support in Ω. $P(D)$ is called normal hyperbolic if it is normal and $P_m(y_1,y^0)$ has m real distinct roots for all $y^0\neq 0$. P is called hyperbolic if it can be obtained from a normal hyperbolic operator by an automorphism of R^n.

The author gives four conditions which are equivalent to $P(D)$ being hyperbolic. Two are: (a) $P(D)$ equidominate $\{D^p\}$, $p\leq m-1$; (b) there exists an $h\in R^n$ and an A such that $|y+ih|^{m-1}\leq A|P_m(y+ih)|$ for all $y\in R^n$. He also considers hyperbolic operators which depend C^∞ on a parameter v. For these he shows that we can choose elementary solutions which behave nicely in v. Finally, some of the results are extended to variable coefficient equations. *L. Ehrenpreis* (Waltham, Mass.)

5364:

Trèves, François. Domination et opérateurs paraboliques. C. R. Acad. Sci. Paris **246** (1958), 867–870.

Notations same as in preceding review except that now m denotes the degree of $P(x,y)$ in y_1. $P(x,D)$ will be assumed normal. We set $P_1(x,y)=(2\pi i)^{-1}(\partial/\partial y_1)P(x,y)$; for each $p\geq 0$ we define $P_{m,p}(x,y)$ as the part of $P(x,y)$ such that $P_{m,p}(x,y_1{}^p,y^0)$ is homogeneous of degree mp. $P(D)$ is called p-parabolic if there exists a $p\geq 0$ so that $P(y_1{}^p,y^0)$ is of degree mp and, for y^0 on the unit sphere, the imaginary parts of the zeros of $P_{m,p}(y_1,y^0)$ in y_1 stay larger than some fixed $\sigma>0$.

The author gives four conditions equivalent to p-parabolicity. The first two are: (a) $P(D)$ equidominates the D^r for $pr_1+r_2+\cdots+r_n\leq mp-1$ according to the base of domination constituted by $\exp(-hx_1)$, $h>0$ [see his paper, same C. R. **245** (1957), 1200–1203; MR **19**, 755]; (b) there exists an A such that for all $y\in R^n$ and all $h\geq 0$, we have

$$(|y^0|^p+|y_1-ih|)^m\leq A|P_{m,p}(y_1-ih,y^0)|.$$

The author considers parabolic operators which depend C^∞ on a parameter v. For these there exists an elementary solution which depends nicely on v. Finally, some of the results are extended to $P(x,D)$.

{There is a misprint in the formula of Theorem 2; the right side should be δ_x.} *L. Ehrenpreis* (Waltham, Mass.)

POTENTIAL THEORY

See also 5432, 5433.

5365:

Aržanyh, I. S. Representation of the bi-wave vector by potentials with double retardation. Dokl. Akad. Nauk Uzbek. SSR **1955**, no. 8, 3–7. (Russian. Uzbek summary)

5366:

Rvačov, V. L. On the solution of a problem in potential theory. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 144–146. (Ukrainian. Russian and English summaries)

"A method is proposed for the approximate solution of the problem of obtaining a harmonic function that is constant on a given surface (S_0) and vanishes at infinity." *Author's summary*

5367:

Zmorovič, V. A. On a generalization of Poisson's integral formula for n-connected circular domains. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 698–701. (Ukrainian. Russian and English summaries)

"In this note the author establishes a new form of the generalized integral formula of Poisson which differs from those given by H. Meschkowski [Ann. Acad. Sci. Fennicae. Ser. A I. Math.-Phys. no. 166 (1954); MR **15**, 695] and O. Lokki [ibid. no. 144 (1952); MR **14**, 742] and is more convenient for application. Employing this formula, the author proves some general theorems about the integral representation of different classes of analytic and harmonic functions in n-connected circular domains. In particular, a generalized Poisson-Jensen formula is established." *Author's summary*

5368a:

Nikol'skiĭ, S. M. Boundary properties of functions defined on a region with angular points. II. Harmonic functions on rectangular regions. Mat. Sb. N.S. 43(85) (1957), 127–144. (Russian)

5368b:

Nikol'skiĭ, S. M. Boundary properties of functions defined on a region with angular points. III. Connection with polyharmonic problems. Mat. Sb. N.S. 45(87) (1958), 181–194. (Russian)

The author continues his study, begun in Mat. Sb. N.S. 40(82) (1956), 303–318 [MR 18, 723], of the boundary properties of functions defined on a region with angular points. As before, the results consist mainly of inequalities between L_p norms of functions on the region and their corresponding boundary functions, suitable assumptions being made as to membership in various of the Nikol'skiĭ classes. The results are too involved to summarize here.

M. G. Arsove (Seattle, Wash.)

5369:

Židkov, G. V. Boundary properties of differentiable and harmonic functions in regions containing angular points. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 225–227. (Russian)

Let G be a conical region with boundary Γ, and let f be the solution of the Dirichlet problem on G for boundary function φ. A number of theorems are stated which ensure that one of the functions φ, f belongs to a Nikol'skiĭ class whenever the other does.

M. G. Arsove (Seattle, Wash.)

5370:

Možerova, N. I. Boundary properties of harmonic functions in three-dimensional space. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 636–638. (Russian)

Given a solution u of the Dirichlet or Neumann problem for the boundary function f over a domain D, to what extent are differentiability properties of f transferred to u? Generalizing the classical answers to this question (obtained by Kellogg and Lichtenstein) the author shows that u belongs to one of the Nikol'skiĭ classes provided f belongs to a correspondingly complicated class of boundary functions. *M. G. Arsove* (Seattle, Wash.)

5371:

Gol'šteĭn, E. G. Estimates on the derivatives of harmonic polynomials in several variables. Akad. Nauk Armyan. SSR Dokl. 26 (1958), 193–200. (Russian. Armenian summary)

Theorem: In p-dimensional space let D be a Jordan region having a smooth boundary; if $P_n(x_1, x_2, \cdots, x_p)$ is a harmonic polynomial of degree n and M is a constant such that $|P_n(q)|\leq M$ $(q \in D)$, then to each $\varepsilon>0$ and each positive integer k there corresponds a positive number c, independent of n, such that

$$\left|\frac{\partial^k P_n(q)}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_p^{k_p}}\right|\leq cMn^{k+\varepsilon}\quad (q\in D),$$

where $k_1+k_2+\cdots+k_p=k$. Under suitable further restrictions on D the result holds with $\varepsilon=0$.

M. G. Arsove (Seattle, Wash.)

5372:

Bugrov, Ya. S. Properties of polyharmonic functions. Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), 491–514. (Russian)

Extensive work on the Dirichlet problem for functions in the Nikol'skiĭ classes is extant in the current Russian literature. This is greatly ramified, and interconnected to a degree that makes it rather difficult to pick up the thread of the development in a single article. The present paper is an exception, since it is largely self-contained and includes a particularly clear, detailed presentation of the important aspects of the theory. Bugrov's methods, direct and elegant, also yield generalizations of results of Nikol'skiĭ and Amanov for the harmonic and polyharmonic Dirichlet problems for the unit circle.

M. G. Arsove (Seattle, Wash.)

5373:

Beurling, A.; et Deny, J. Espaces de Dirichlet. I. Le cas élémentaire. Acta Math. 99 (1958), 203–224.

Die vorliegende Arbeit bildet die Einführung in Untersuchungen über eine Klasse von Funktionenräumen, für welche eine Potentialtheorie ohne "Kern" entwickelt werden soll. Eine grundlegende Rolle spielen lineare Abbildungen ("Kontraktionen") $T(z)$ in der komplexen Ebene C, welche für je zwei Zahlen z_1 und z_2 den Abstand verringern: $|T(z_1)-T(z_2)|\leq|z_1-z_2|$ und normal genannt werden, wenn der Punkt 0 fest bleibt. Ist X ein lokal kompakter Hausdorffscher Raum, auf dem ein positives überall dichtes Mass ξ existiert, $\mathfrak{C}$ der Raum aller komplexwertigen, stetigen Funktionen mit kompaktem Träger und $D=D(X,\xi)$ ein vollständiger Hilbertraum von komplexwertigen Funktionen, welche auf X f.ü. (in Bezug auf ξ) definiert sind und auf jeder kompakten Teilmenge ξ-integrierbar sind, so wird $D(X,\xi)$ Dirichletscher Raum genannt, wenn gilt: (1) Für jede kompakte Teilmenge $K\subseteq X$ existiert eine Zahl $A(K)$, sodass

$$\int_K |u(x)|d\xi(x)\leq A(K)\|u\|$$

für alle $u\in D$; (2) $\mathfrak{C}\sim D$ ist dicht in $\mathfrak{C}$ und D; (3) für jede normale Kontraktion T und jedes Element $u\in D$ ist $T(u)\in D$ und $\|T(u)\|\leq\|u\|$.

Der klassische Dirichletsche Raum wird erhalten, indem man die Klasse aller in einem Gebiet $\omega\subseteq R^m$ definierten, komplexwertigen, unendlich oft differenzierbaren Funktionen mit kompaktem Träger durch

$$\|u\|^2=\int_\omega |\text{grad}\, u(x)|^2dx$$

normiert und in geeigneter Weise abschliesst. Der so gewonnene Dirichletsche Raum $\hat{D}_\omega$ kann bekanntlich durch Greensche Potentiale endlicher Energie erzeugt werden. Diese können aber auch ohne Greenschen Kern definiert werden; es sind die Funktionen $u\in\hat{D}_\omega$, für welche ein Radonsches Mass μ auf ω existiert, mit

$$(u,\varphi)=\int\bar{\varphi}d\mu \text{ für alle } \varphi\in\mathfrak{C}\sim\hat{D}_\omega.$$

Man nennt dann u das durch μ erzeugte Potential. Diese Definition überträgt sich ohne weiteres auf allgemeine Dirichletsche Räume. Der lineare Operator, der einer Funktion $u\in D$ das Mass μ zuordnet, wird als verallgemeinerter Laplace-Operator bezeichnet. Eine Theorie der Balayage, des Gleichverteilungsproblems und ein bisher in der klassischen Theorie unausgenutztes "principe des condensateurs" wird angekündigt.

In diesem ersten Teil wird ein Raum X von n Punkten x_i $(i=1, 2, \cdots, n)$ betrachtet. Bezeichnet $\mathfrak{C}$ den Vektorraum aller komplexwertigen Funktionen u auf X, so werden zur Erzeugung eines Hilbertraumes Hermitesche Formen $H(u)$ herangezogen, welchen eine auf $X\times X$ definierte Funktion $\Delta(x,y)$ zugeordnet werden kann, mit deren Hilfe der verallgemeinerte Laplace-Operator gebildet wird. Gilt für jede normale Kontraktion $T(u)$: $H(T(u))\leq H(u)$, so heißt $H(u)$ eine Dirichletsche Form. Ist $H(u)$ positiv definit, so heißt $H(u)^{\frac{1}{2}}$ eine Dirichletsche Norm und $\mathfrak{C}$, mit dieser Struktur versehen, ein Dirichletscher Raum. Für diese Dirichletschen Formen bzw. Räume werden zahlreiche potentialtheoretische Charakteri-

sierungen gegeben. Eine Theorie des Dirichletschen Problems wird entwickelt. Den Schluss bilden drei sehr instruktive Beispiele aus Elektrodynamik, Elektrostatik und der Theorie der konformen Abbildung.

K. Endl (Giessen)

5374:

Duffin, R. J.; and Shelly, E. P. Difference equations of polyharmonic type. Duke Math. J. **25** (1958), 209–238.

Cette étude concerne les fonctions définies sur un réseau (ensemble de points de R^n à coordonnées entières). L'opérateur harmonique D est défini, à partir des opérateurs de translation T_i, par $D=\sum_{i=1}^{n}(T_i+T_i^{-1})-2nI$; les fonctions p-harmoniques sont les solutions de $D^pu=0$. Poursuivant une étude antérieure de l'un d'eux [R. J. Duffin, même J. **23** (1956), 335–363; MR **1**, 1193], les A. établissent pour ces fonctions diverses propriétés analogues à celles des fonctions polyharmoniques: (a) Les fonctions p-harmoniques définies sur un domaine convexe peuvent être prolongées indéfiniment, et admettent une représentation de la forme $w(x)=\sum_{i=0}^{p-1}x_1^ih_i$, où x_1 désigne la première coordonnée de x, et $\{h_i\}$ un système de p fonctions harmoniques. (b) Il existe une fonction de Green g_p (définie par une intégrale de Fourier), satisfaisant à $D^pg_p=0$ sauf en un point "source" où $D^pg_p=(-1)^p$; pour $n=2$ et $n=3$ la méthode de Duffin permet d'obtenir le comportement asymptotique de g_p; pour $n=2$ on construit, à partir de g_1, une autre fonction de Green biharmonique h, qui ne diffère de g_2 que par un multiple de x^2+y^2, et des tables de valeurs de g_1 et h sont données; pour $n=3$, $g_1(x, y, z)$ est de la forme $A\lambda_1+B\lambda_2+C\lambda_3$, les constantes A et C étant connues exactement, et B approximativement, et des tables de valeurs des fonctions rationnelles λ_1, λ_2, λ_3 (présumées harmoniques) sont données. (c) Les auteurs étudient ensuite des opérateurs plus généraux, dits de type p-harmonique. Si L est un tel opérateur, et u une solution de $Lu=0$, bornée sur une boule de rayon R, alors les différences d'ordre m de u au centre sont majorées par une quantité de la forme AC_mR^{2p-m}, d'où il résulte que si on a partout $u(x)=O[1+\|x\|^{\eta/2}]$, avec $\eta>0$, alors u est un polynome de degré $\leq\eta$. Ces résultats, ainsi que l'étude des polynomes polyharmoniques qui les suit, généralisent une étude de Heilbronn [Proc. Cambridge Philos. Soc. **45** (1949), 194–206; MR **10**, 705].

J. Lelong (Paris)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 5374.

5375:

Talacko, Joseph. Some operational methods in the calculus of finite differences. Math. Mag. **31** (1957/58), 15–25.

Some known formulas relating operators in the calculus of finite differences are derived by the symbolic method. {There are a number of errors and omitted definitions.}

J. Riordan (New York, N.Y.)

SEQUENCES, SERIES, SUMMABILITY

See also 5282.

5376:

Milkman, Joseph. Logarithmic sequences. Proc. Amer. Math. Soc. **8** (1957), 1114–1124.

For $n>1$, a sequence $a_1, \cdots, a_n$ is "logarithmic" if $0\leq a_i\leq 1$, $a_i+a_{i'}=a_j+a_{j'}$ for $ii'=jj'$, $\Delta=\min_{ii'<jj'}(a_j+a_{j'}-a_i-a_{i'})>0$. In this system of linear inequalities Δ is maximized by a variant, developed by the author and J. F. Paydon, of the Fourier descent method [see the review MR **16**, 558, Šteĭnberg]. It is conjectured and for $n\leq 7$ shown that $\delta=1/\max\Delta$ is an integer (1, 3, 6, 16, 23, 42), with unique $a_1, \cdots, a_n$. {Reviewer verified this for $8\leq n\leq 13$, with $\delta=54$, 82, 136, 159, 208, 266. So far $\Delta=2a_{n-1}-a_n-a_{n-2}$. Preferably replace the condition $a_i\leq 1$ by $\Delta\geq 1$; then $\delta=\min a_n$.}

T. S. Motzkin (Los Angeles, Calif.)

5377:

Zmorovič, V. A. On some questions of the theory of convergence of positive numerical series. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 805–809. (Ukrainian. Russian and English summaries)

"The author proves some theorems on the tests for convergence and divergence derived by N. I. Lobachevsky, V. P. Yermakov, A. Cauchy, O. Schlömilch and N. V. Bugayov for positive numerical series."

Author's summary

5378:

Tyler, Barbara. Absolute convergence and summability factors in a sequence. J. London Math. Soc. **33** (1958), 341–351.

A sequence $s_0, s_1, \cdots$ is said to be evaluable $|C, k|$, or absolutely evaluable (C, k), if its Cesàro transform $\sigma_0, \sigma_1, \cdots$ of order k is the sequence of partial sums of an absolutely convergent series. The following theorem is proved. If k and ρ are non-negative integers, then a given factor sequence ε_n is such that the sequence ε_ns_n is evaluable $|C, \rho|$ whenever the sequence s_n is evaluable $|C, k|$, if and only if (i) $\varepsilon_n=O(n^{\rho-k})$, (i') $\sum_{p=0}^{n}\varepsilon_p=O(n)$, (ii) $\Delta^{k-1}\varepsilon_n=O(n^{1-k})$ and (iii) $\sum_{n=0}^{\infty}|\Delta C^\rho(\varepsilon_n)|<\infty$. The single theorem covers all three cases $\rho=k$, $\rho>k$, and $\rho<k$. The results and methods are different from, but somewhat similar to, those of Bosanquet and others involving Cesàro evaluability of series of the form $\sum\varepsilon_nu_n$.

R. P. Agnew (Ithaca, N.Y.)

5379:

Kangro, G. On the generalization of a theorem of Moore. Dokl. Akad. Nauk SSSR **121** (1958), 967–969. (Russian)

Let $A=(a_{nk})$ be a real matrix with a $a_{kk}\neq 0$, $A^{-1}=(a_{nk}')$. Let

$$D_\nu=\sup_k|a_{\nu+k,\nu+k}, a'_{\nu+k,k}|.$$

The coefficients ε_k are called convergence factors of the method A if $\sum\varepsilon_kx_k$ converges for each A-summable series $\sum x_k$. If $\sum\nu D_\nu<+\infty$, then ε_k are convergence factors of A if and only if: (1) $\sum_{k=0}^{\infty}\varepsilon_k\sum_{\nu=0}^{k}a_{k\nu}'$ converges; (2) $|\varepsilon_k|=O(a_{kk})$; (3) $\sum_{k=0}^{\infty}|\sum_{\nu=k}^{\infty}a_{\nu k}'\varepsilon_\nu|<+\infty$. There are several related theorems, also for the case when x_k are elements of a Banach space X and ε_k operators mapping X into itself.

G. G. Lorentz (Syracuse, N.Y.)

5380:

Vermes, P. Summability of power series in simply or multiply connected domains. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 188–199.

J. Teghem [C. R. du Congrès de l'A.F.A.S., Caen, 1955; Colloque sur la théorie des suites, Brussels, 1957, pp. 87–95, Gauthier-Villars, Paris, 1958] has made a survey of results of Borel type relating to the analytic continuation of a power series by matrix transformations. All these results are based on the domain of summability D of a matrix applied to the series $\sum z^k$, and it has been proved

under various conditions on the matrix or also on a function $f(z)$ that the matrix is efficient for a general power series $\sum u_k z^k$ representing $f(z)$ in the Mittag-Leffler star corresponding to D. It was observed by Teghem that in all these results D is a simply-connected open domain, so that its complement $C(D)$ is a connected closed subset of the infinite z-plane. He suggested the problem of finding examples of matrices for which D is multiply connected and mentioned that results of Borel type might perhaps be extended to this case.

In the present paper, various examples of matrices are constructed where: (a) D is multiply connected, so that $C(D)$ is not connected; (b) D is simply connected, but $C(D)$ consists of two or more discrete two-dimensional regions connected by a curve or having just one point in common, or one or both of the regions is redueed to a point. In all but one of the constructions, use is made of a theorem of Walsh on polynomial approximations to a regular function, viz. [J. L. Walsh, Mém. Sci. Math. Fasc. 73 (1935); p. 7]: Let R be an arbitrary bounded closed point set, and let $f(z)$ be analytic at every point of R. Then $f(z)$ can be uniformly approximated on R by a polynomial in z if, and only if, R does not separate any singularity of $f(z)$ from the point at infinity. Using this theorem, it is shown that Borel type results cannot be applied in case (a) above, but that they do apply to cases of type (b) if the Mittag-Leffler star is generalized to a curvilinear star-domain. In case (a), the following theorem is proved. If $g_n(z)=\sum_{k=0}^{\infty} g_{n,k}z^k$ converges for $n=0, 1, 2, \cdots$, and if $g_n(z)$ tends to the finite limit $1/(1-z)$ in a multiply connected domain D, and does not tend to a limit outside D, then the convergence to $1/(1-z)$ cannot be uniform in every finite closed region inside D.

R. G. Cooke (London)

5381:

Orudžev, Gardašhan. Convergence of Newton's interpolation series. Trudy Azerbaĭdžan. Gos. Ped. Inst. Lenin. 2 (1955), 146–153. (Russian)

For the Newton series

$$(*)\quad \sum_{n=1}^{\infty} a_n(z-x_1)\cdots(z-x_n),\ x_n=S_n/P_n,\qquad P_1+\cdots+P_n=S_n\to\infty,$$

under the assumption $\sum (P_n/S_n)^\mu<+\infty$, $\mu\leq 2$, the following results are proved: (1) if (*) converges at $z=z_0$, then it converges for all z with $\mathrm{Re}(z-z_0)>\mu-1$; (2) series (*) and the Dirichlet series

$$\sum_{n=1}^{\infty} c_n \exp(-\lambda_n z),\ c_n=(-1)^n a_n S_1\cdots S_n P_1^{-1}\cdots P_n^{-1},\qquad \lambda_n=\log S_n,$$

converge and diverge at the same time.

G. G. Lorentz (Syracuse, N.Y.)

5382:

Spitkovskaya, K. M. Remainder terms of certain power series. Izv. Kiev. Politehn. Inst. 16 (1954), 243–252. (Russian)

APPROXIMATIONS AND EXPANSIONS

See also 5157, 5294, 5371, 5427.

5383a:

Abdullaev, I. K. Best approximation in the mean to functions of the form $\int_0^\infty |x|^s d\psi(s)$ on the interval $[-1; +1]$. Trudy Azerbaĭdžan. Gos. Ped. Inst. Lenin. 2 (1955), 97–109. (Russian)

5383b:

Abdullaev, I. K. Best (uniform) approximation by polynomials of a function of the form

$$\psi_{s,m}(a-x)=(a-x)^{s/2}[b-\ln(a-x)]^{-m}.$$

Trudy Azerbaĭdžan. Gos. Ped. Inst. Lenin 2 (1955), 181–187. (Russian)

5383c:

Ahundov, A. Best approximation in the mean to the function

$$F_{s,m}(|a-x|)=|a-x|^s\ln^m|a-x|.$$

Trudy Azerbaĭdžan. Gos. Ped. Inst. Lenin. 2 (1955), 117–132. (Russian)

In these papers, methods of S. Bernstein [Leçons sur les propriétés extrémales, Gauthier-Villars, Paris, 1926; Izv. Akad. Nauk SSSR, Ser. Mat. **10** (1946), 185–196; MR **8**, 267], Nikol'skiĭ [same Izv. **11** (1947), 139–180; MR **8**, 576] and Ibragimov [same Izv. **10** (1946), 429–460; MR **8**, 459] are used to obtain asymptotic expressions for the degree of approximation E_n of functions in question on $(-1, 1)$ by polynomials in the metric of the spaces L^1 and C. We list some of the formulas. In the first paper:

$$E_n(|x|^s(b-\log|x|)^{-m})_{L^1}\approx M_s n^{-s-1}\log^{-m} n,$$

where $s>0$ is not an even integer and

$$M_s=8\pi^{-1}|\sin\tfrac{1}{2}\pi s|\int_0^\infty|\cos v|dv\times\int_0^\infty u^{-s-1}(u^2+v^2)^{-1}(e^u+e^{-u})^{-1}du.$$

In the second paper:

$$E_n(\psi_{s,m})_C\approx\pi^{-1}|\sin\tfrac{1}{2}\pi s|\Gamma(\tfrac{1}{2}s+1)(a^2-1)^{s-2}(a+\sqrt{(a^2-1)})^{-n}n^{-\frac{1}{2}s-1}\log^{-m} n,$$

where $a>1$, $s>0$ is not an even integer; in the third paper

$$E_n(F_{s,m})_{L^1}\approx M_s(1-a^2)^{\frac{1}{2}s+1}n^{-s-1}\log^m n,$$

where $|a|<1$, $s>-1$ is not an even integer, m is a positive integer.

G. G. Lorentz (Syracuse, N.Y.)

5384a:

Guseĭnov, G. A. Approximation of discontinuous functions by generalized Bernstein polynomials. Trudy Azerbaĭdžan. Gos. Ped. Inst. Lenin. 2 (1955), 133–145. (Russian)

5384b:

Guseĭnov, G. A. Approximation of summable, semicontinuous and measurable functions by generalized Bernstein polynomials. Trudy Azerbaĭdžan. Gos. Ped. Inst. Lenin. 2 (1955), 163–180. (Russian)

The author gives generalizations of known theorems about Bernstein polynomials [Lorentz, Bernstein polynomials, Univ. of Toronto Press, 1953; MR **15**, 217; see pp. 30–34, 46–48] to polynomials

$$(*)\qquad B_n(x)=\sum_{k=0}^{n} f(\sigma_{nk})p_{nk}(x)$$

defined by Hirschman and Widder and Gelfond and to similar polynomials in two variables. The behavior of $B_n(x)$ in the points of discontinuity of the first kind is discussed; the expression $f(\sigma_{nk})$ in (*) is replaced by the integral mean, the metric mean, the maximum of $f(x)$ in a small interval around σ_{nk}, and results stated which

correspond to those of Kantorovič and the reviewer. {Although the results announced are essentially correct, the formulations of the theorems and the proofs are unreliable. For example, the proofs of Theorems 5 and 6 of the first paper are illusory, and the inequality (3), p. 134, used throughout, is taken over from a paper of Gelfond [Izv. Akad. Nauk SSSR Ser. Mat. 14 (1950), 413–420; MR 12, 332] with misprints contained there.}
G. G. Lorentz (Syracuse, N.Y.)

5385:

Bojanić, R. Sur l'approximation des fonctions continues par les polynômes de Bernstein. Glas Srpske Akad. Nauka 232 Od. Prirod.-Mat. Nauka (N.S.) 15 (1958), 59–65. (Serbo-Croatian. French summary)

If $f(x)$ has a continuous derivative on $[0, 1]$ except at $x=\xi$, where $f'(x)$ has the jump σ, then for the Bernstein polynomial $B_n(x)$ of $f(x)$,

$$B_n(\xi)-f(\xi)=[\xi(1-\xi)/2\pi n]^{\frac{1}{2}}\sigma+o(n^{-\frac{1}{2}}).$$

This is obtained by finding an asymptotic expression for

$$\sum_{k=0}^{n}\binom{n}{k}\xi^k(1-\xi)^{n-k}|\xi-k/n|.$$

G. G. Lorentz (Syracuse, N.Y.)

5386:

Zuhovickiĭ, S. I. An algorithm for constructing the Chebyshev polynomial approximation to a continuous function. Dokl. Akad. Nauk SSSR 120 (1958), 693–696. (Russian)

Let Q be a compact metric space, $f(q)$ and $\varphi_1(q), \cdots, \varphi_n(q)$ be given real continuous functions on Q, $x(q)=\sum_1^n \xi_k\varphi_k(q)$ polynomials in φ_k, $\Delta(x, q)=x(q)-f(q)$. The author gives a method for obtaining in a finite number of steps, for a given $\eta>0$, a compact subset Q' of Q and a polynomial x_0 such that Q' is an η-net in Q and that

$$\max_{q\in Q'}|\Delta(x_0, q)|\leqq\inf_x \max_{q\in Q}|\Delta(x, q)|.$$

The completion of each step requires the following operations. For a given x, the points are determined for which $|\Delta(x, q)|$ attains its maximum; for a given x and points $q_1, \cdots, q_p$ of Q, with

$$|\Delta(x, q_1)|=\cdots=|\Delta(x, q_p)|,$$

the relative gradient $z=\sum \zeta_k\varphi_k(q)$ is determined; that is z for which

$$\frac{d}{d\varepsilon}|\Delta(x+\varepsilon z, q)|_{\varepsilon=0}$$

is negative and maximal in absolute value, subject to the conditions

$$\sum \zeta_k^2=\text{const}; |\Delta(x+\varepsilon z, q_1)|=\cdots=|\Delta(x+\varepsilon z, q_p)|.$$

This z can be found from a system of linear equations.
G. G. Lorentz (Syracuse, N.Y.)

5387:

Berman, D. L. On the impossibility of constructing a linear polynomial operator furnishing an approximation of the order of the best approximation. Dokl. Akad. Nauk SSSR 120 (1958), 1175–1177. (Russian)

Let E be a space of 2π-periodic functions of the (Köthe-Toeplitz) type described by the author in his previous papers [same Dokl. 85 (1952), 13–16; 88 (1953), 9–12; MR 14, 57, 767]. Let T_n denote trigonometric polynomials of degree $\leqq n$, $E_n(f)=\inf_{T_n}\|f-T_n\|$, $S_n(f)$ the partial sum of the Fourier series of f. Assume that $\|f-S_n(f)\|$ is unbounded for some $f\in E$. Using his former results, the author proves that no sequence of continuous linear operators $U_n(f)$ mapping E into itself can have the properties: 1) For each f, $U_n(f)$ is a T_n; 2) $\|U_n(f)-f\|=O(E_n(f))$ for each $f\in E$. In particular, E can be the space C or L^r.
G. G. Lorentz (Syracuse, N.Y.)

5388:

Burkill, J. C. Polynomial approximations to functions with bounded differences. J. London Math. Soc. 33 (1958), 157–161.

Soit $f(x)$ une fonction de la variable réelle x, continue dans un segment, $h_0, h_1, \cdots, h_n$ désignant $(n+1)$ points différents quelconques de ce segment, posant:

$$\varphi(u)=(u-h_0)\cdots(u-h_n),\ D_n(f; h_0, \cdots, h_n)=\sum_{i=0}^{n}\frac{f(h_i)}{\varphi'(h_i)},$$

$$T_n=\sum_0^n\frac{1}{|\varphi'(h_i)|},$$

il existe un polynome $p_{n-1}(x)$, de degré $(n-1)$ au plus, tel que:

$$|f(x)-p_{n-1}(x)|\leqq\sup\frac{D_n}{T_n}(f).$$

Lorsqu'il s'agit du segment $[-1, +1]$, on a: $T(h_0, \cdots, h_n)\geqq 2^{n-1}$, l'égalité n'ayant lieu que si les points h_i sont ceux où $\cos(n \arccos x)$ prend les valeurs ±1.
J. Favard (Paris)

5389a:

Berg, Lothar. Über das asymptotische Verhalten der Laplace-Transformation. Wiss. Z. Hochschule Elektrotechn. Ilmenau 2 (1956), 77–78.

5389b:

Berg, Lothar. Über das asymptotische Verhalten der Laplace-Transformation (Fortsetzung). Math. Nachr. 17 (1958), 57–61.

Let $f(s)$ be the Laplace transform of $F(t)$. In a general way one expects the behavior of F at 0 to be reflected by the behavior of f at ∞. The author gives the following theorems in this direction. (First paper) Let $F_0(t)$, $\Phi_0(t)$, $\rho(t)$ be continuous near 0, with $\Phi_0(t)>0$ and $\rho(t)\geqq0$, but $\rho(t)$ not identically zero in any $(0, \delta)$. If $F(t)=F_0(t)\rho(t)$, $\Phi(t)=\Phi_0(t)\rho(t)$, then $F_0(t)\sim\Phi_0(t)$ as $t\to0$ implies $f(s)\sim\phi(s)$ as $s\to\infty$. (Second paper) If $F_0(t)=o(\Phi_0(t))$ then $f(s)=o(\phi(s))$. Next, allow F_0 and Φ_0 to have complex values; let ρ be continuous near 0; let $\Phi_0(t)$ be zero-free near 0. Let the Laplace transform $\phi^*(s)$ of $|\Phi(t)|$ exist and satisfy $\phi^*(s)=O(|\phi(s)|)$, $s\to\infty$. Then $F_0(t)\sim A\Phi_0(t)$ $(t\to0)$ implies $f(s)\sim A\phi(s)$ $(s\to\infty)$. Corollary: if $\Phi(t)$ is continuous near 0 and positive for $t>0$, and F, Φ, G have Laplace transforms, then $F(t)=G(t)+o(\Phi(t))$ for $t\to0$ implies $f(s)=g(s)+o(\phi(s))$ for $s\to\infty$. The author notes that $F\sim\Phi$ does not always imply $f\sim\phi$. By combining the preceding theorem with the results of the paper reviewed below, the author obtains the following result: if among the functions $\chi(t)$ for which $F(t)\sim e^{-\chi(t)}$ $(t\to0)$ there is one, say $\chi_0(t)$, satisfying the set of hypotheses stated below, then

$$f(s)\sim(2\pi)^{\frac{1}{2}}\{\chi_0''(x)\}^{-\frac{1}{2}}\exp(-sx-\chi_0(x)),$$

where x is the solution of $s+\chi_0'(x)=0$. The conditions on χ are: $\chi''(x)$ continuous for x near 0; there is an $\omega(x)$, $0<\omega<x$, with $\omega^2\chi''(x)\to+\infty$ and $\chi''(\xi(x))\sim\chi''(x)$ as $x\to0$ whenever $|\xi(x)-x|\leqq\omega(x)$.
R. P. Boas, Jr. (Evanston, Ill.)

5390:

Berg, Lothar. Asymptotische Darstellungen für Integrale und Reihen mit Anwendungen. Math. Nachr. 17 (1958), 101–135.

The author presents a general version of the Laplace method for asymptotic evaluation of integrals, with many applications. Earlier investigations along the same lines were reported in Z. Angew. Math. Mech. 36 (1956), 245–246; same Nachr. 16 (1957), 195–205 [MR 18, 207; 20 #94]. Extensions were announced in Z. Angew. Math. Mech. 37 (1957), 260–261.

Let $g(s, t)$ have continuous first and second derivatives with respect to t for $t \geqq t_1$; the derivatives are denoted respectively by g_1 and g_2. Let $g_1(s, x)=0$ have the solution $x=x(s)$. The central result is as follows. Let there exist a function $\omega(s)$ such that for $s \to S$ (S may be, and usually is, ∞),

$$x(s)-t_1 \geqq \omega(s)>0;\ \omega^2(s)g_2(s, x(s)) \to +\infty;$$
$$g_2(s, \xi(s)) \sim g_2(s, x(s))$$

whenever $|x(s)-\xi(s)| \leqq \omega(s)$. Let

$$g_2(s, x)^{\frac{1}{2}} \int_a^{x-\omega} \exp(-g(s,t)+g(s,x))dt \to 0$$

(and the same with $\int_{x+\omega}^b$). Then, as $s \to S$,

$$(*) \qquad \int_a^b e^{-g(s,t)}dt \sim (2\pi)^{\frac{1}{2}}[g_2(s, x(s))]^{-\frac{1}{2}} e^{-g(s,x(s))}.$$

Less complicated and frequently applicable cases are given, for example, the following. If $g_2(s, x+\omega) \sim g_2(s, x)$ for every ω with $\omega=o(x)$ and if $g(s, t) \geqq g(s, x-\omega)$ for $a \leqq t \leqq x-\omega$, $g(s, t)-2 \log t \geqq g(s, x+\omega)-2 \log(x+\omega)$ for $t \geqq x+\omega$, and $x^2 g_2(s, x) \to +\infty$, then (*) holds. An alternative theorem, which seems to be of a different character, applies when $g_2(s, x) \sim c/x^2$, $c \neq 0$. The applications include many familiar and unfamiliar theorems about Mellin transforms, Laplace transforms, and series, with extensions to complex functions and complex domains.

R. P. Boas, Jr. (Evanston, Ill.)

5391:

van der Corput, J. G. On the coefficients in certain asymptotic factorial expansions. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 60=Indag. Math. 19 (1957), 337–351.

Suppose that $f(z)$ possesses the following properties in the sector $-\pi+\varepsilon<\arg z<\pi-\varepsilon$ with $\varepsilon>0$. First, there exist three numbers a, b and c such that $f(z)e^{-az}z^{-bz-c} \to L \neq 0$ as $\operatorname{Re} z \to \infty$, where L is independent of the (fixed) value of $\operatorname{Im} z$. Second, $f(z+1)/f(z)$ admits an asymptotic expansion $\psi_0(z)+\psi_1(z)+\cdots$ for large $|z|$, where $\psi_0(z)$, $\psi_1(z)$, $\cdots$ are rational functions of z which approach zero as $|z| \to \infty$. Then

$$f(z)=A^z\alpha^{\alpha z}\left\{\sum_{m=0}^{M-1} c_m/\Gamma(\alpha z+\beta+m)+O(1/\Gamma(\alpha z+\beta+M))\right\},$$
$$1/f(z)=A^{-z}\alpha^{-\alpha z}\left\{\sum_{m=0}^{M-1} \gamma_m\Gamma(\alpha z+\beta-m)+O(\Gamma(\alpha z+\beta-M))\right\},$$

where A, α and β are easily found from $\psi_0(z)$ and $\psi_1(z)$, and where c_m and γ_m are given by recurrence relations.

T. E. Hull (Pasadena, Calif.)

FOURIER ANALYSIS

See also 5431.

5392:

Ivašev-Musatov, O. S. On coefficients of trigonometric null-series. Izv. Akad. Nauk SSSR. Ser. Mat. 21 (1957), 559–578. (Russian)

A trigonometric null-series (t.n.s.) is a series

$$\sum a_n e^{inx} \quad (a_n=\bar{a}_{-n},\ \sum |a_n|>0),$$

which converges to zero for almost all x. Such a series must satisfy

$$(*) \qquad \sum |a_n|^2=\infty.$$

The author proves that (*) may diverge very slowly, in the following sense.

Theorem 1: Let $\chi(y)$ be a continuous function defined on $y \geqq 0$, such that: 1) $\int_0^\infty \chi^2(y)dy=\infty$; 2) $y\chi^2(y) \to 0$ for $y \to \infty$; 3) $y^{1+\varepsilon}\chi^2(y) \to 0$ for $y \to \infty$ and for any $\varepsilon>0$; 4) there exists an $m>\frac{1}{2}$ such that $y^m\chi(y)$ is non-decreasing; then there exists a t.n.s. such that $a_n=o(\chi(|n|))$.

The proof is based on the following lemma: If $\chi(y)$ is as above, there exists a non-decreasing singular function $F(x)$ defined on $[0, 2\pi]$ such that F is constant on each contiguous interval of a certain perfect set of measure zero, and such that

$$\int_0^{2\pi} e^{-iyx}dF(x)=o(\chi(|y|)).$$

A similar result is proved for non-decreasing singular functions defined on $(-\infty, \infty)$: Theorem 2: If $\chi(y)$ is as above, there exists a continuous non-decreasing singular function $F(x)$ on $(-\infty, \infty)$, such that $F(x) \to 0$ for $x \to -\infty$, $F(x) \to 1$ for $x \to \infty$, $F(x)$ is constant on each contiguous interval of certain perfect set of measure zero, and

$$\int_{-\infty}^{\infty} e^{-iyx}dF(x)=o(\chi(|y|)).$$

M. Cotlar (Buenos Aires)

5393:

Timan, M. F. Converse theorems in the constructive theory of functions of many variables. Dokl. Akad. Nauk SSSR 120 (1958), 1207–1209. (Russian)

Ein Beweis der Ungleichung:

$$\|\Delta_{h_1}^{r_1} \Delta_{h_2}^{r_2} f(x, y)\| \leqq \frac{C}{m^{r_1}n^{r_2}} \sum_{k=1}^{m} \sum_{l=1}^{n} k^{r_1-1}l^{r_2-1}E_{k-1,l-1}$$

wo

$$E_{k,l}=E_{k,l}f=\inf \|f(x, y)-T_{k,l}(x, y)\|;$$

$h_1=O(1/m)$, $h_2=O(1/n)$; $T_{m,n}(x, y)$ — ein trigonometrisches Polynom der Ordnung m nach x und der Ordnung n nach y; und

$$\Delta_{h_1}{}^{r_1}\Delta_{h_2}{}^{r_2}f(x, y)=$$
$$\sum \sum (-1)^{r_1+r_2-i-j}C_{r_1}{}^iC_{r_2}{}^jf(x+ih_1, y+jh_2).$$

$f(x, y)$ ist eine periodische Funktion der Periode 2π nach jeder der Veränderlichen x und y.

B. Germansky (Berlin)

5394:

Freud, Géza; and Ganelius, Tord. Some remarks on one-sided approximation. Math. Scand. 5 (1957), 276–284.

Soit H_r la classe de fonctions d'une variable réelle, de période 2π, primitives d'ordre r d'une fonction à variation bornée; pour tout n, il existe un polynome trigonométrique W_n d'ordre n, tel que:

$$V_n=W_n-h \geqq 0, \quad \|W_n-h\| \leqq \frac{C_r}{n^{r+1}} \int_{-\pi}^{+\pi} |dh^{(r)}|,$$

où C_r désigne une constante qui ne dépend que de r. On a, uniformément en θ:

$$W_n^{(q)}(\theta)-h^{(q)}(\theta)=O(n^{q-r}) \quad (0 \leqq q \leqq r);$$

en un point de continuité de $h^{(r)}$, O est à remplacer par o.

Ce résultat est localisé dans le suivant: si, dans un intervalle I, on a: $h(\theta)=\eta(\theta)$, où $\eta(\theta)\in H_s$, avec $s>r$, alors on a, dans tout segment $I^*\subset I$:

$$\int_{I^*} V_n(\theta)d\theta=O(n^{-s-1}),\int_{I^*} |dV_n|=O(n^{-s}).$$

J. Favard (Paris)

5395:

Theiler, G. A direct demonstration of Kronecker's theorem in the theory of almost periodic functions and some of its consequences. Lucrarile Inst. Petrol Gaze București 4 (1958), 275–291. (Romanian. Russian and English summaries)

Expository article. *R. P. Boas, Jr.* (Evanston, Ill.)

5396:

Lefranc, Marcel. Analyse spectrale sur Z_n. C. R. Acad. Sci. Paris **246** (1958), 1951–1953.

The author considers the space E of complex-valued functions of n variables $x_1, \cdots, x_n$, where each x_r ranges through the additive group of integers. Call an element of E an exponential polynomial if it is of the form

$$Q(x_1, \cdots, x_n)\xi_1^{x_1}\cdots\xi_n^{x_n},$$

where Q is a polynomial and the ξ_r are non-zero complex numbers. It is proved that, if E has the topology of pointwise convergence, every closed translation-invariant vector subspace A of E, other than zero and E itself, is the closed subspace generated by its exponential polynomials. The dual of E is considered as a ring of polynomials in n indeterminates and the proof depends on the primary decomposition of the ideal orthogonal to A.

A. P. Robertson (Glasgow)

5397:

Helson, Henry. Conjugate series and a theorem of Paley. Pacific J. Math. **8** (1958), 437–446.

Let T_k $(1\leqq k\leqq\aleph_0)$ be the product of k copies of the circle group and $\mathscr{L}_k$ the character group of T_k. The group $\mathscr{L}_k$ is representable as the additive group of all integer-valued lattice points in k-dimensional real space having only a finite number of non-zero co-ordinates. A half-space S in $\mathscr{L}_k$ is a subsemigroup of $\mathscr{L}_k$ not containing 0 such that $N\in\mathscr{L}_k$ is in S if and only if $-N$ is not. For a function $f\in L_1(T_k)$ with Fourier series

$$(1)\qquad \sum_{N\in\mathscr{L}_k}\hat{f}(N)e^{iN\cdot X},$$

let T_S be the operator carrying the series (1) into the series

$$(2)\qquad \sum_{N\in S}\hat{f}(N)e^{iN\cdot X}.$$

Theorem: Let $0<p<1$. If $f\in L_1(T_k)$, then the series (2) is summable in the topology of $L_p(T_k)$ to a limit function T_Sf, for which $\|T_Sf\|_p\leqq A_p\|f\|_1$. The summation is effected by any approximate identity in $L_1(T_k)$ consisting of trigonometric polynomials. Theorem: For $k<\aleph_0$, let $\{w(N)\}_{N\in S}$ be a non-negative function on S such that

$$\sum_{N\in S}|\hat{f}(N)|w(N)<\infty$$

for every continuous function f on T_k whose Fourier transform vanishes off S. Then $\sum_{N\in S}w(N)^2<\infty$. This result generalizes a theorem of Paley [J. London Math. Soc. **7** (1932), 122–130]. The analogue for $k=\aleph_0$ is false, as shown by H. Bohr [Nachr. Akad. Wiss. Goettingen, Math.-Phys. Kl. **1913**, 441–488; see p. 468]. For other results concerning half-spaces, see the author's and Lowdenslager's recent joint paper [Acta Math. **99** (1958), 165–202; MR **20** #4155]. *E. Hewitt* (Seattle, Wash.)

5398:

Watari, Chinami. On generalized Walsh Fourier series. I. Proc. Japan Acad. 33 (1957), 435–438.

Let H_k be the group of integers modulo $\alpha(k)$, $k=1, 2, \cdots$, taken with the discrete topology; then $H=\prod_{k=1}^{\infty}H_k$ is an infinite compact group. If $\alpha(k)=2$, $k=1, 2, \cdots$, then there is a familiar mapping of H onto the unit interval in which the characters of H become the Walsh functions. In the present note the author treats the case where the $\alpha(k)$ are any uniformly bounded sequence. [The case $\alpha(k)=\alpha$ $(k=1, 2, \cdots)$ has been studied by Chrestenson, Pacific J. Math. **5** (1955), 17–31; MR **16**, 920.] The author lists for this case a large number of theorems, most of them analogues of results obtained by R. E. A. C. Paley [Proc. London Math. Soc. **34** (1932), 241–279]. *I. I. Hirschman, Jr.* (St. Louis, Mo.)

INTEGRAL TRANSFORMS

See also 5389a-b, 5390, 5432, 5433.

5399:

Sarymsakov, T. A. The method of moments and sequences of polynomials with regular distribution of zeros. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1954**, no. 6, 91–98. (Russian. Uzbek summary)

5400:

Solov'ev, A. F. Generalization of a theorem of Hausdorff. Uspehi Mat. Nauk 13 (1958), no. 6(84), 167–171. (Russian)

The author gives necessary and sufficient conditions for the μ_n in order that the moment problem $\int_0^1 x^n f(x)dx=\mu_n$, $n=0, 1, \cdots$, have a solution $f(x)$ belonging to an Orlicz space X of measurable functions on $(0, 1)$. The same result was obtained by the reviewer [Lorentz, "Bernstein polynomials", Univ. of Toronto Press, 1953; MR **15**, 217; p. 80] for the more general case when X is any space of measurable functions on $(0, 1)$ with the property that the norm $\|f\|$ is invariant under equimeasurable rearrangements of $f(x)$.

G. G. Lorentz (Syracuse, N.Y.)

5401:

Kumar, Ram. On generalised Hankel-transform. I, II. Bull. Calcutta Math. Soc. **49** (1957), 105–118.

The author investigates the integral transformation whose kernel is

$$(xy)^{\gamma}\sum_{k=0}^{\infty}\frac{(-\frac{1}{4}x^2y^2)^k}{k!\Gamma(1+\lambda+\mu k)}.$$

If $f(x)$ is the transform of $g(y)$, $\mu^{-1}y^{-(\lambda+1)+\mu(\gamma+1)}g(y^{\mu})$ is the Laplace transform of $t^{(\alpha+1)\mu-1}\psi(t^{\mu})$, and $t^{\varrho}\psi(t^{-1})$ is the Hankel transform of order ν of $s^{\nu}\phi(s)$, then, under certain assumptions on the parameters, he expresses $f(x)$ as an integral involving $\phi(s)$, gives two corollaries of this result and applies it to evaluate several integrals. The second paper contains a result of a similar character with attendant examples. *A. Erdélyi* (Pasadena, Calif.)

5402:

Koizumi, Sumiyuki. On the singular integrals. I. Proc. Japan Acad. 34 (1958), 193–198.

Results on Hilbert's operator in the n-dimensional Euclidean space E^n are generalised. A. P. Calderón and A. Zygmund [Acta Math. **88** (1952), 85–139; MR **14**, 637]

had extended the Riesz theory, defining the operator

$$\text{(I)} \qquad \bar{f}(P)=\int_{E^n} K(P-Q)f(Q)dQ,$$

where P, Q are points in E^n and the kernel K satisfies a number of conditions. This is their main result: If $f\in L^p(E_n)(1<p<\infty)$, then $\bar{f}\in L^p$ also, and

$$\text{(II)} \qquad \int_{E^n}|\bar{f}(P)|^p dP\leqq A\int_{E^n}|f(Q)|^p dQ,$$

where the constant A depends on K, p and n only. The author generalises this and some results related to it. He introduces a class L^φ such that $f(Q)$ is measurable and $\varphi(|f|)$ integrable; where $\varphi(u)$ is a continuous increasing function, $\varphi(0)=0$, $\varphi(2u)=O(\varphi(u))$, $\int_u^\infty \varphi(t)t^{-r-1}dt=O(u^{-r}\varphi(u))$ $(1<r<\infty)$, $\int_1^u \varphi(t)t^{-2}dt=O(u^{-1}\varphi(u))$ as $u\to\infty$, and where there are similar conditions for $u\to 0$. The function $u^p=\varphi(u)$, where $1<p<r$, is a special case. Using the Markiewicz-Zygmund results on interpolation of operations, he deduces an auxiliary theorem on a quasilinear operation and finally the inequality

$$\text{(II}') \qquad \int_{E^n}\varphi(|\bar{f}|)dP\leqq A\int_{E^n}\varphi(|f|)dQ;$$

and results related to this. *H. Kober* (Birmingham)

5403:

Koizumi, Sumiyuki. On the singular integrals. II. Proc. Japan Acad. 34 (1958), 235–240.

From their results on Hilbert's operator E^n $(n>1)$ Calderón and Zygmund [Studia Math. 14 (1954), 249–271; MR 16, 1017] had derived a theory of the periodic case, which is analogous to the conjugate-function operator

$$\bar{f}(\varphi)=(2\pi)^{-1}\int_{-\pi}^{\pi} f(t)\cot\tfrac{1}{2}(\varphi-t)dt$$

by using a modified kernel (cf. review above) $K^*(Q)$. Here the author introduces a function $\varphi(u)$, defined for $u\to\infty$ as above and, in addition, a

$$\chi^*(u)=u\int_1^u t^{-2}\chi(t)dt,$$

where $\chi(t)$ is a positive increasing function and

$$\chi(u)=0\ (u\leqq 1),\quad \chi(2u)=O(\chi(u)),$$

$$\int_u^\infty \chi(t)t^{-r-1}dt=O(\chi(u)u^{-r})\ (1<r<\infty;\ u\to\infty).$$

He considers the classes L and L^{χ^*} of measurable functions such that $\varphi(|f|)$ and $\chi^*(|f|)$, respectively, are integrable over the fundamental cube R, i.e., over $|\xi_i|\leqq\frac{1}{2}$, $i=1, 2, \cdots$, where ξ_i are the coordinates of a point Q. Using a similar method as in his previous paper he proves that

$$\int_R \varphi(|\bar{f}^*|)dP\leqq A\int_R \varphi(|f|)+B,$$

$$\int_R \chi(|\bar{f}^*|dP\leqq A\int_R \chi^*(|f|)dQ+B,$$

where $\bar{f}^*(P)$ is the transform of a periodic function $f(Q)$. Thus basic results of Zygmund and Calderón are generalised, since $L^\varphi=L^p$ for $\varphi=u^p$, $1<p<r$, and since $\chi^*=u\log u$ when $\chi(u)=u$, $u>1$.

In a similar way, the author generalises results about the discrete analogues of Hilbert transforms.

H. Kober (Birmingham)

5404:

Saksena, K. M. Inversion and representation theorems for a generalization of Laplace transformation. Nieuw Arch. Wisk. (3) 6 (1958), 1–9.

This is an investigation of the transformation

$$F(x)=\int_0^\infty e^{-\frac{1}{2}xt}(xt)^{-k-\frac{1}{2}}W_{k+\frac{1}{2},m}(xt)\varphi(t)dt,$$

where $W_{k,m}$ denotes the Whittaker function. It was introduced by Meijer as a generalisation of the Laplace integral; it reduces to this for $k=-m$. Meijer obtained a complex inversion formula by generalising the corresponding result on the Laplace transformation. Erdélyi has expressed the kernel as a fractional integral of e^{-xt}. Here, a real inversion formula is deduced, analogous to that constructed by Widder for the Laplace integral, with the aid of the fractional-integration operator

$$K_{\zeta,\alpha}{}^- f=(\Gamma(\alpha))^{-1}x^\zeta\int_x^\infty (u-x)^{\alpha-1}u^{-\zeta-\alpha}f(u)du.$$

Defining

$$U_0 f=f(x),\ U_q f=(-1)^q x^{q-k-m}(d/dx)^q[x^{k+m}f(x)],$$
$$V_{q,u}f=(u\Gamma(q))^{-1}[U_q f]_{x=q/u}\ (q=1, 2, \cdots),$$

he proves that

$$\phi(u)=\lim V_{q,u}F\ (q\to\infty;\ u>0),$$

provided that $\phi(u)$ is bounded for $0<u<\infty$; and he deduces necessary and sufficient conditions for a given $F(x)$ to be representable by Meijer's integral, with $\phi(t)$ bounded. *H. Kober* (Birmingham)

5405:

Pollard, Harry; and Widom, Harold. Inversion of an integral transform. Proc. Amer. Math. Soc. 9 (1958), 598–602.

Inversion formulae are given for the integral transformation

$$f(x)=\int_0^\infty k(y)\phi(x+y)dy\ (x>0)$$

under the conditions that $k\in L(0,\infty)$; $k\not\equiv 0$ near 0; $\phi\in L_p(0,\infty)$, $1\leqq p\leqq 2$; $K(s)=\int_0^\infty e^{-sx}k(x)dx\neq 0$ $(\Re s\geqq 0)$. The conclusion then is that

$$\phi(x)=\underset{\delta\to 0+}{\text{l.i.m.}}\lim_{\varepsilon\to 0+}\frac{1}{2\pi}\int_0^\infty f(x+y)dy\int_{-\infty}^\infty \frac{e^{it(y-\delta)-\varepsilon|t|}}{K(it)}dt,$$

the limit in mean being in $L_p(0,\infty)$, and

$$\phi(x)=\lim_{\delta\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{2\pi}\int_0^\infty f(x+y)dy\int_{-\infty}^\infty \frac{1-e^{-i\delta t}}{i\delta t}\frac{e^{ity-\varepsilon|t|}}{K(it)}dt$$

for almost all x and, in particular, at all points of right-continuity of ϕ. *H. R. Pitt* (Nottingham)

5406:

Rao, V. V. L. N. Self-reciprocal functions in the form of series. Proc. Indian Acad. Sci. Sect. A. 48 (1958), 263–268.

The author obtains formally some expansion formulae for functions self-reciprocal with respect to Hankel transforms. Convergence of the expansions is not studied.

P. G. Rooney (Toronto, Ont.)

5407:

Rao, V. V. L. N. Some self-reciprocal functions and kernels. Proc. Cambridge Philos. Soc. 55 (1959), 62–65.

The author deduces that certain functions are self-reciprocal with respect to Hankel transforms.

P. G. Rooney (Toronto, Ont.)

5408:

Ogievec'kiĭ, I. I. Generalization of P. Civin's inequality for the fractional derivative of a trigonometric polynomial to L_p-space. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 486–488. (Ukrainian. Russian and English summaries)

Let $A_k(x)=a_k \cos kx+b_k \sin kx$ and $B_k(x)=a_k \sin kx-b_k \cos kx$. For the trigonometric polynomial $T_n(x)=\frac{1}{2}a_0+\sum_{k=1}^n A_n(x)$, one defines the fractional derivative of order σ by

$$T_n{}^\sigma(x)=\cos \tfrac{1}{2}\pi\sigma \sum_{k=1}^n k^\sigma A_k(x)-\sin \tfrac{1}{2}\pi\sigma \sum_{k=1}^n k^\sigma B_k(x).$$

If one denotes by $\|\cdot\|_p$ the L^p norm on $[0, 2\pi]$, $1\leqq p\leqq\infty$, the inequality obtained is

$$(*) \qquad \|T_n{}^\sigma\|_p\leqq C(\sigma)n^\sigma\|T_n\|_p,$$

where $C(\sigma)$ is a constant depending only on σ. The inequality that is generalized is the case $p=\infty$ of $(*)$ [Duke Math. J. 8 (1941), 656–665; MR 3, 108].

P. Civin (Eugene, Ore.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 5356, 5357, 5362.

5409:

Sahnovič, L. A. Spectral analysis of operators of the form $Kf=\int_0^\infty f(t)k(x-t)dt$. Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), 299–308.

The article gives sufficient conditions for the existence of a root of nth degree of the operator $Kf=\int_0^\infty f(t)k(x-t)dt$. As corollaries, certain theorems are obtained on the reduction of K to simplest form. *Author's summary*

5410:

Yanovskiĭ, S. V. Some questions connected with equations of convolution type. Rostov. Gos. Ped. Inst. Uč. Zap. 4 (1957), 79–88. (Russian)

The equations studied are of the form

$$\sum \Big\{\lambda_{ijk}f_j^{(k)}(x)+\int_0^\infty k_{ijk}'(x-t)f_j^{(k)}(t)dt +\int_{-\infty}^0 k_{ijk}''(x-t)f_j^{(k)}(t)dt+\int_{-\infty}^\infty k_{ijk}(x-t)f_j^{(k)}(t)dt\Big\}=g_i(x).$$

Results are stated concerning the classes of numbers such that $f_j^{(k)}(t)e^{\alpha t}$ are summable over $(0, \infty)$ and $(-\infty, 0)$ on the supposition that the k belong to classes of functions for which corresponding integrals exist, and that the integrals involved in the equation converge.

J. L. B. Cooper (Cardiff)

FUNCTIONAL ANALYSIS

See also 5259, 5351, 5352, 5368a-b, 5396, 5409.

5411:

✱Taylor, Angus E. Introduction to functional analysis. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xvi+423 pp. $12.50.

This book is primarily intended "to assist graduate students in learning about linear spaces and linear operators . . . and their unifying power in problems of algebra, classical analysis, the theory of integration, and differential and integral equations; it is also intended to be useful to mathematicians, both pure and applied".

There are seven chapters, with the titles: Abstract approach to linear problems; Topologies; Topological linear spaces; General theorems on linear operators; Spectral analysis of linear operators; Spectral analysis in Hilbert space; Integration and linear functionals.

In the first chapter the approach is purely algebraic. Linear operators in finite dimensional spaces are fully illustrated and then used as helpful analogues throughout the book. Many important examples are given for operators in infinite-dimensional spaces, e.g., the operators occurring in integral equations of Fredholm types. Next come linear functionals, the algebraic conjugate of a linear space, and then the basic theorems on extensions of linear operators and functionals, all still in purely algebraic form. The chapter closes with those theorems on the connection between the range (null manifold) of an operator and the null manifold (range) of its transpose which, in the finite-dimensional case, include the standard results for systems of linear algebraic equations.

After a concise presentation, in Chapter Two, of basic reference material from general topology, Chapter Three deals with normed linear spaces, including Banach spaces, inner-product spaces, Hilbert spaces. The latter part of the chapter takes up general linear spaces, though chiefly with a view to the further study of normed spaces, which are emphasized throughout the book. It is in this setting that the author discusses convex sets and linear varieties The chapter closes with material on weak topologies, duality, and metric spaces.

Chapter Four, which begins the study of linear operators, falls naturally into three parts. The first part introduces the algebra of continuous linear operators from one linear space to another, together with such concepts as boundedness and closedness of an operator, the uniform topology, etc., and culminates with the interior mapping principle and the closed-graph theorem. The second part deals with conjugate spaces and operators, beginning with the Hahn-Banach theorem and continuing with several basic theorems on uniform boundedness of various kinds. The possible relations between a linear operator and its conjugate are summed up in a "state-diagram" requiring 81 squares, since a priori each of the two operators may be in any one of 9 states according to whether its range is the whole of the range-space, dense in it, or not dense, and whether its inverse is continuous, non-continuous or non-existent. The last part of the chapter deals with inner-product spaces, in particular with the representation theorem for continuous functionals and the projection theorem.

Chapter Five, concerned with closed, but not necessarily continuous, linear operators, begins with the basic theorems about resolvent operators and sets, continuous and residual spectra and eigenvalues. It deals chiefly with completely continuous (compact) operators, for whose spectra the chief theorems are given. The chapter continues with a remarkably clear account of the operational calculus, based on contour integration in the complex plane and here used to prove the basic reducibility theorems about spectral sets and projections, the spectral mapping theorem, and theorems on the nature of the resolvent near an isolated point of the spectrum.

Chapter Six presents the standard material on self-adjoint, normal and unitary operators in Hilbert space and gives a clear, detailed treatment of the spectral theorem for bounded self-adjoint operators and for

unitary operators. For unbounded self-adjoint operators, which play so great a role in quantum mechanics, the situation is sketched without proofs.

The seventh chapter presents in outline a modern generalization of Lebesgue measure and integration, the classical form of which is sufficient for the earlier chapters of the book. However, this final chapter is related to the rest of the book, not only in that it clarifies the representation theorems for linear functionals, but also because its material is essential for further study of the more recent developments of spectral theory in the general setting of Banach algebras.

The problems distributed throughout the book are very numerous and helpful; illustrative examples are given everywhere, many of them explaining at considerable length the applications of the subject; in every chapter there are illuminating summaries at the beginning and at the end; and each chapter has a copious bibliography. In his preface the author hopes that the book will "open doors for the student ... to the new frontiers of modern mathematics ... with a clearer realization of the structure of classical mathematics." The book is well adapted to fulfill this hope.

S. H. Gould (Providence, R.I.)

5412:

Makarov, B. M. Inductive limits of normed spaces. Dokl. Akad. Nauk SSSR **119** (1958), 1092–1094. (Russian)

Earlier results of Sebastião e Silva [Rend. Mat. e Appl. (5) **14** (1955), 388–410; MR **16**, 1122] and Raĭkov [Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 984–986; MR **19**, 754] are generalized. Eight theorems are announced without proofs, of which the following are typical. The projective (inductive) limit of a sequence of reflexive Banach spaces is a reflexive space, and its conjugate space is an inductive (projective) limit of reflexive Banach spaces. The inductive limit E of a sequence E_n of locally convex spaces with Hausdorff separation also has Hausorff separation, and every set bounded in E is contained in some E_n. Other theorems relate to closed subspaces and factor spaces. *E. Hewitt* (Seattle, Wash.)

5413:

Rolewicz, S. On inversion of non-linear transformations. Studia Math. **17** (1958), 79–83.

Let X and Y be F-spaces (complete metric linear spaces) and let U be a continuous one-one mapping of X onto Y. It is noted that, if U is non-linear, the inverse function U^{-1} can be discontinuous at a dense set of points. It is proved that a map U of a metric space into a separable F-space is of the first Baire class if and only if it is of the first Borel class [C. Kuratowski, *Topologie, I*, Monograf. Mat. vol. 20, Warsaw-Wroclaw, 1948; MR **10**, 389; p. 280].

Let X, Y be F-spaces with X separable. Let τ, σ denote modes of convergence in X, Y, respectively, and suppose that $U:X\to Y$ is one-one and maps τ-convergent sequences into σ-convergent sequences. Further, suppose that each ball in X is sequentially τ-compact and that norm convergence in Y implies σ-convergence. Then the inverse map U^{-1} is of the first Baire class relative to norm convergence. *R. G. Bartle* (Urbana, Ill.)

5414:

Gavurin, M. K. On the fundamental theorems of the differential and integral calculus in linear spaces. Vestnik Leningrad. Univ. **13** (1958), no. 7, 38–48. (Russian. English summary)

Suppose X is a real linear manifold (no topology assumed) and Y is a multinormed linear space, that is a linear space with a set of non-Hausdorff norms all of which vanish simultaneously only at zero. If f is a function from X to the space of linear operators from X to Y then f', the Gateaux differential of f, can be considered as a bilinear operator. If the differential converges uniformly on every closed segment of X, then the Riemann integral of f around every closed polygon vanishes if and only if f is the differential of some function if and only if f' is symmetric. *D. C. Kleinecke* (Livermore, Calif.)

5415:

Edwards, R. E. Derivatives of vector-valued functions. Mathematika **5** (1958), 58–61.

By use of the Banach-Steinhaus theorem and a lemma on real-valued functions with locally Lipschitzian derivatives, the author proves the following: Let E be a quasi-complete separated locally convex space, and let f be a function mapping the real axis R into E. If $p=0$ is an integer, and if each scalar component of f admits a locally Lipschitzian pth derivative, then f has derivatives of all orders $\leqq p$ in the sense of the initial topology of E, and these derivatives are locally Lipschitzian. (For each x' in the topological dual E' of E, the corresponding scalar component of f is the function $f_{x'}$ on R to R given by $f_{x'}(r)=x'(f(r))$ for each $r\in R$.) This result is closely related to a differentiability theorem of L. Schwartz [Ann. Inst. Fourier Grenoble **2** (1950), 17–18; MR **13**, 138], although neither includes the other.

V. L. Klee, Jr. (Copenhagen)

5416:

Lane, Ralph E. Linear operators on quasi-continuous functions. Trans. Amer. Math. Soc. **89** (1958), 378–394.

A function $y(t)$ on $-\infty<t<+\infty$ is quasi-continuous if $y(t+)$ and $y(t-)$ exist for every t. The operators T, called Q operators on an interval $[a, b]$, are transformations on the class of quasi-continuous functions to functions on $-\infty<s<\infty$ which are linear and satisfy the conditions: (a) if $z(t)=y(t+c)$, then $Tz(s)=y(s+c)$; and (b) for every s there exists $B_s\geqq 0$ such that if y is quasi-continuous and $M>|y(s-t)|$ for $a\leqq t\leqq b$, then $Ty(s)\leqq MB_s$. The latter condition defines a norm $|T(s)|$, which turns out to be $|T(0)|$. Such a T transforms quasi-continuous functions into quasi-continuous functions, functions of bounded variation into functions of bounded variation and continuous functions into continuous functions. It is shown that for a Q operator T on $[a, b]$ there exist two functions $x_1(t)$ and $x_2(t)$ such that if $y(t)$ is quasi-continuous then

$$Ty(s)=\int_a^{b+} g(s-t)dx_1(t)+\int_{a-}^b h(s-t)dx_2(t),$$

where g is left continuous and h is right continuous and $y(s)=g(s)+h(s)$. Presumably, the integrals are Stieltjes mean integrals based on successive subdivisions. [See Lane, Proc. Amer. Math. Soc. **5** (1954), 59–66; MR **15**, 514.] In case T has the additional property that

$$Ty(s+)-Ty(s-)=2[Ty_R(s)-Ty_L(s)],$$

where $y_R(t)=y(t+)$ and $y_L(t)=y(t-)$ for all t, then

$$Ty(s)=\int_a^b y(s-t)dx(t),$$

where $x(t)$ is of bounded variation on $[a, b]$, vanishes for $t<a$ and is equal to $x(b)$ for $t>b$. Conditions that the resulting functions of such operations have certain properties are derived. The paper concludes with an operator

such that polynomials of degree less than 2 are invariant, and Ty has an nth derivative for all quasi-continuous functions, which has proved valuable as a smoothing operator in connection with the computation of integrals.
T. H. Hildebrandt (Ann Arbor, Mich.)

5417:

Andô, Tsuyoshi. On the continuity of norms. Proc. Japan Acad. 33 (1957), 429–434.

The author presents some improvements of results of H. Nakano [*Modulared semi-ordered linear spaces*, Maruzen, Tokyo, 1950; MR 12, 420] concerning the continuity of the norm in a universally continuous normed semi-ordered linear space R. (The norm is continuous if $a_n \downarrow 0$ implies $\inf \|a_n\|=0$.)

Suppose that every segment of R is complete. If, in addition, (a) the norm is equally monotone (there exists $\delta>0$ such that $a \cap b=0$, $\|a\|=\|b\|=1$ imply $\|a+b\| \geqq 1+\delta$) or (b) the norm on R is locally uniformly convex or (c) there exists $\delta>0$ such that $\|(|a|+\delta a)\|+\|(|a|-\delta a)\| \leqq 2+\delta$ for all a, $\|a\| \leqq 1$, then the norm on R is continuous. Other sufficient conditions are also given.
B. Yood (New Haven, Conn.)

5418:

Gelbaum, Bernard R. Conditional and unconditional convergence in Banach spaces. An. Acad. Brasil. Ci. 30 (1958), 21–27.

Let G_∞ be the topological product of a sequence of copies of the two-element group $\{-1, 1\}$. For $g \in G_\infty$, let $\varepsilon_n(g)$ denote the nth coordinate of g. If $y=\{y_n\}$ is a sequence of elements in a Banach space E, let $y(g)=\sum_1^\infty \varepsilon_n(g) y_n$ when the latter exists. In case E has a basis $\{x_n; X_n\}$ [B. R. Gelbaum, Duke Math. J. 17 (1950), 187–196; MR 11, 729], an element $x \in X$ is identified with the sequence $\{X_n(x)x_n\}$. The author studies topological and (Haar) measure-theoretic properties of the set $\{g : y(g)$ exists$\}$ for a given sequence y, and various related sets in G_∞. For $x \in E$, he considers the set $R(x)$ of all $x(g)$ that exist and proves that the following are equivalent: (a) $R(x)$ is compact for all x; (b) $R(x)$ is weakly compact for all x; (c) $\{x_n; X_n\}$ is an unconditional basis for E.
M. Jerison (Princeton, N.J.)

5419:

Gelbaum, Bernard R. Notes on Banach spaces and bases. An. Acad. Brasil. Ci. 30 (1958), 29–36.

As a contribution to the problem of existence of a basis, it is shown that every separable infinite-dimensional Banach space E contains a dense, linearly independent sequence $\{x_n\}$ such that for each $x \in E$ there exists a sequence of scalars $\{b_n(x)\}$ satisfying $\sum b_n(x)x_n=x$. Let $E_1=\{\{a_n\} : \sum_1^\infty a_n x_n$ exists$\}$ with norm given by $\sup_m \|\sum_1^m a_n x_n\|$, and $N_1=\{\{a_n\} : \sum_1^\infty a_n x_n=0\}$. Then N_1 is a closed subspace of the Banach space E_1. The coefficients $b_n(x)$ may be chosen as continuous linear functionals on E if and only if N_1 is complemented in E_1.
M. Jerison (Princeton, N.J.)

5420:

Grünbaum, Branko. Two examples in the theory of polynomial functionals. Riveon Lematematika 11 (1957), 56–60. (Hebrew. English summary)

R. S. Martin showed that if f^* is the polar form of a homogeneous polynomial f of degree n on a Banach space, then $\|f^*\| \leqq \|f\| n^n/n!$. In the first example, $f(x)$ is the product of the coordinates of the point x in n-dimensional l_1. Then $\|f\|=n^{-n}$, while $\|f^*\|=1/n!$. This shows that Martin's bound is the best possible, notwithstanding an assertion of A. D. Michal [Math. Mag. 27 (1954), 119–126; MR 15, 630]. (The validity of Michal's proof is questioned in the review.) The second example consists of the Euclidean plane X embedded in a 3-dimensional space Y and a homogeneous quadratic polynomial functional on X that has no extension to a homogeneous polynomial on Y with the same norm. The polar of this functional is, then, a bilinear functional on $X \times X$ having no extension to a bilinear functional on $Y \times Y$ with the same norm.
M. Jerison (Princeton, N.J.)

5421:

Fan, Ky; and Glicksberg, Irving. Some geometric properties of the spheres in a normed linear space. Duke Math. J. 25 (1958), 553–568.

It is possible to define in terms of sequences in a normed space E a number of "rotundity" properties of E intermediate between strict convexity and uniform convexity of E [Clarkson, Trans. Amer. Math. Soc. 40 (1936), 396–414]. The authors list here a number of such sequential properties, some familiar. They show how the properties are related in general normed spaces and also in complete normed spaces, and show which of the properties imply reflexivity of E.

This paper concludes with a discussion of the relation of these properties to such earlier concepts as the uniform convexity of Clarksen, the local uniform convexity of Lovaglia [Trans. Amer. Math. Soc. 78 (1955), 225–238; MR 16, 596], and the full k-convexity of the authors' earlier note [Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 947–953; MR 17, 386].
M. M. Day (Urbana, Ill.)

5422:

Kadec, M. I. On strong and weak convergence. Dokl. Akad. Nauk SSSR 122 (1958), 13–16. (Russian)

It is well-known that in any uniformly convex Banach space, if the sequence (x_n) converges weakly to x_0 and if $|x_n| \to |x_0|$, then the sequence actually converges strongly to x_0. This paper shows that in any separable Banach space there is an equivalent norm relative to which it enjoys this property. Since $C[0, 1]$ is a universal space for separable Banach spaces, it suffices to show that $C[0, 1]$ enjoys the stated property relative to the norm defined by

$$|f|=\sup\{|f(t)| ; t \in [0, 1]\}+\sum_{k=1}^{\infty} 2^{-k}\omega(f, 1/k),$$

where $\omega(f, \delta)=\sup\{|f(t)-f(s)| ; |t-s|<\delta\}$. As a consequence of this theorem it is demonstrated that any two separable reflexive Banach spaces are homeomorphic.
R. G. Bartle (Urbana, Ill.)

5423:

Singer, Ivan. Un dual du théorème de Hahn-Banach. C. R. Acad. Sci. Paris 247 (1958), 408–411.

If x is an element of a normed space E and $\mathcal{M}$ is a subspace of E^*, let $\|x\|_{\mathcal{M}}=\sup\{f(x) : f \in \mathcal{M}$ and $\|f\| \leqq 1\}$. Consider the following approximate dual to the Hahn-Banach theorem: If $x \in E$ and $\varepsilon>0$ there exists $y \in E$ such that $\|y\| \leqq \|x\|_{\mathcal{M}}+\varepsilon$ and $f(x)=f(y)$ for each $f \in \mathcal{M}$. This is well known if $\mathcal{M}$ is finite dimensional; the author proves that it is true if $\mathcal{M}$ is assumed to be regularly (i.e., weak*-, or $\sigma(E^*, E)$-) closed. It is asserted to be true for any $\mathcal{M}$. {As the author notes in the paper reviewed below, this assertion is false even if $\mathcal{M}$ is assumed to be norm-closed.}
R. R. Phelps (Princeton, N.J.)

5424:

Singer, Ivan. Quelques applications d'un dual du théorème de Hahn-Banach. C. R. Acad. Sci. Paris 247 (1958), 846–849.

The author applies his theorem from the paper reviewed

above to a very short proof of a generalization (to certain infinite sets of functionals) of a theorem of Helly on the existence of an element of E at which a finite set of functionals takes on prescribed values. Another application yields an equally short proof of a theorem by Banach on regularly closed subspaces of E^*. {The effect of this brevity is marred by the fact that one third of the paper is devoted to the proof of Lemma 3, which (by using the equivalence between weak* and regular closure of subspaces of E^*) is a special case of an elementary theorem on closed subspaces of a topological vector space.} *R. R. Phelps* (Princeton, N.J.)

5425:

Davis, Chandler. Separation of two linear subspaces. Acta Sci. Math. Szeged 19 (1958), 172–187.

For a closed linear subspace $\mathfrak{P}$ of a Hilbert space $\mathfrak{H}$, let $\tilde{\mathfrak{P}}$ denote the orthogonal complement of $\mathfrak{P}$, P the projection of $\mathfrak{H}$ onto $\mathfrak{P}$, and $\tilde{P}=1-P$. The nullspace of a linear operator A will be denoted by $\mathfrak{N}(A)$. For two projections P, Q in $\mathfrak{H}$, the author introduces the closeness operator $C=C(P,Q)=PQP+\tilde{P}\tilde{Q}\tilde{P}$ and the separation operator $S=S(P,Q)=P\tilde{Q}P+\tilde{P}Q\tilde{P}$, which generalize trigonometric functions in case of 2-space. Assume first $\mathfrak{N}(C)=0$. Then

$$U=U(P,Q)=C^{-\frac{1}{2}}(QP+\tilde{Q}\tilde{P}),$$
$$X=X(P,Q)=C^{-\frac{1}{2}}(P-\tilde{Q}),$$
$$Y=Y(P,Q)=\tfrac{1}{2}(1+X),$$

are well-defined on $\mathfrak{H}$. U is a unitary operator on $\mathfrak{H}$ mapping $\mathfrak{P}$ onto $\mathfrak{Q}$. X is the unique symmetry on $\mathfrak{H}$ which exchanges $\mathfrak{P}$ with $\mathfrak{Q}$ and satisfies $PXP\geq 0$. Y is the projection onto a closed linear subspace $\mathfrak{Y}$ which is, roughly speaking, the angle bisector of the acute angle between $\mathfrak{P}$ and $\mathfrak{Q}$. Several relations such as

$$U(P,Y)=U(Y,Q),\ U(P,Q)=U(P,Y)^2=X(P,Q)(P-\tilde{P})$$

are established. Two closed linear subspaces $\mathfrak{P}$, $\mathfrak{Q}$ are said to be equivalently positioned if $\dim(\mathfrak{P}\cap\tilde{\mathfrak{Q}})=\dim(\tilde{\mathfrak{P}}\cap\mathfrak{Q})$. When $\mathfrak{N}(C)\neq 0$, but $\mathfrak{P}$, $\mathfrak{Q}$ are equivalently positioned, the above formula for U still defines U on $(\mathfrak{N}(C))^{\sim}$. Since $\mathfrak{N}(C)$ is spanned by $\mathfrak{P}\cap\tilde{\mathfrak{Q}}$ and $\tilde{\mathfrak{P}}\cap\mathfrak{Q}$ one can define U on $\mathfrak{P}\cap\tilde{\mathfrak{Q}}$ as some isometry V onto $\tilde{\mathfrak{P}}\cap\mathfrak{Q}$, and define U on $\tilde{\mathfrak{P}}\cap\mathfrak{Q}$ as $-V^*$. Similarly, one modifies the definition of $X(P,Q)$. The main results can be summarized as follows. (I) $\mathfrak{P}$ and $\mathfrak{Q}$ are equivalently positioned if and only if there exists a unitary operator W on $\mathfrak{H}$ which commutes with $C(P,Q)$ and carries $\mathfrak{P}$ onto $\mathfrak{Q}$. Among all unitary W carrying $\mathfrak{P}$ onto $\mathfrak{Q}$, the operator $U(P,Q)$ defined above has certain minimal properties, e.g., $\|1-U\|\leq\|1-W\|$. (II) A complete set of unitary invariants for a pair of closed linear subspaces $\mathfrak{P}$, $\mathfrak{Q}$ consists of the dimensionalities of $\mathfrak{P}\cap\mathfrak{Q}$, $\mathfrak{P}\cap\tilde{\mathfrak{Q}}$, $\tilde{\mathfrak{P}}\cap\mathfrak{Q}$ and $\tilde{\mathfrak{P}}\cap\tilde{\mathfrak{Q}}$ (any 4 cardinal numbers may occur) and a spectral multiplicity function on measures on [0, 1], that of $C(P,Q)$ restricted to $(\mathfrak{N}(C))^{\sim}\cap(\mathfrak{N}(S))^{\sim}$ (it has even values and is zero on the point measures at 0 and 1, but is otherwise arbitrary). (III) An operator A in $\mathfrak{H}$ is the difference of two projections if and only if $-1\leq A\leq 1$ and on $(\mathfrak{N}(1-A^2))^{\sim}$ there exists a unitary W such that $AW=-WA$. (IV) A unitary operator W is the product of two symmetries if and only if its spectrum is symmetric with respect to the real axis. (V) A unitary operator W can be expressed as $W=U(P,Q)$ if and only if its spectrum lies in the closed right half plane and is symmetric with respect to the real axis. Previously, the unitary equivalence problem for a pair $\mathfrak{P}$, $\mathfrak{Q}$ has been studied by J. Dixmier [Revue Sci. **86** (1948), 387–399; MR **10**, 546]. However, the development of the present paper is quite different, and it emphasizes the idea of generalized trigonometry throughout. {On line 8 of page 179, "$\mathfrak{B}$" should be replaced by "$\mathfrak{N}$". In the displayed matrix for $U(P,Y)$ on p. 182, "F" should be replaced by "$\tilde{F}$" in each off-diagonal entry.} *Ky Fan* (Notre Dame, Ind.)

5426:

deLeeuw, Karel; and Rudin, Walter. Extreme points and extremum problems in H_1. Pacific J. Math. 8 (1958), 467–485.

The following fact was proved by Beurling [Acta Math. **81** (1949), 239–255: MR **10**, 381]. A function $f\in H_p$ ($p>0$, $f\neq 0$) has a unique factorization $f=M_fQ_f$, where $M_f(z)$ has the form

$$B(z)\exp\left\{-\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)\right\}$$

($B(z)$ is a Blaschke product and μ is a singular measure) and $Q_f(z)$ has the form

$$c\exp\left\{\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,h(\theta)d\theta\right\}$$

($c\neq 0$ and $h\in L_1(-\pi,\pi)$). The authors prove the following theorems. The extreme points of the unit ball of H_1 are exactly the functions f such that $\|f\|=1$ and $f=Q_f$. If $f\in H_1$, $\|f\|=1$, and f is not an extreme point, then $f=\frac{1}{2}(f_1+f_2)$, where f_1 and f_2 are extreme points. The norm closure of the set of extreme points of the unit ball is the class of all $f\in H_1$ such that $\|f\|=1$ and $f(z)\neq 0$ for $|z|<1$. These facts are applied to a number of closure and extremum problems in H_1. For example, for $f\in H_1$ and $f\neq 0$, the functions $p(z)f(z)$ (p polynomial in z) are dense in H_1 in the norm or weak* topology if and only if $f=Q_f$. This is an analogue of a theorem of Beurling for H_2 [loc. cit.]. Results are also obtained concerning the set where bounded linear functionals on H_1 assume their norms on the unit ball. Finally, the extreme points of the unit ball of H_∞ are identified (a joint effort of five persons). *E. Hewitt* (Seattle, Wash.)

5427:

Krylov, A. L. Boundary problems and biorthogonal expansions in Banach spaces. Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 865–867. (Russian)

The author outlines a method for finding solutions for boundary value problems in $L_p(\Omega)$ by decomposing them into parts to be found by use of Green's function, Neumann's function, and the theorem of Calderón and Zygmund. *M. M. Day* (Urbana, Ill.)

5428:

de Possel, René. Les équations linéaires dans un espace vectoriel algébrique topologique et de Banach. Rev. Math. Pures Appl. 1 (1956), no. 3, 203–207.

This is a survey of results about linear transformations in topological vector spaces which were obtained by the present author and M. Audin and which were announced previously by M. Audin [C. R. Acad. Sci. Paris **237** (1953), 511–512; **238** (1954), 2221–2222; MR **15**, 233; **16**, 142]. *W. A. J. Luxemburg* (Pasadena, Calif.)

5429:

Rădulescu, M. Calculus with entire linear operators. I. Linear derivations of the first order. Lucrarile Inst. Petrol Gaze Bucureşti 4 (1958), 293–336. (Romanian. Russian and English summaries)

In this expository paper the author gives various (algebraic) properties of certain classes of linear operators having as domain and range vector subspaces of complex-valued functions defined on an interval of the real line.

5430:

Lidskiĭ, V. B. Theorems on the completeness of a system of characteristic and adjoint elements of operators having a discrete spectrum. Dokl. Akad. Nauk SSSR 119 (1958), 1088–1091. (Russian)

If T is a Hilbert-Schmidt kernel, $T=A+iB$, where A and B are selfadjoint operators of definite sign, then the characteristic and adjoint elements for the nonzero points of the spectrum of T are complete in the range of T. If L_1 and L_2 are symmetric operators on a domain D dense in the Hilbert Space, if for some λ the set $(L_1+iL_2-\lambda E)D$ is dense in the space, and if L_1 and L_2 are semibounded and one of the selfadjoint extensions of L_1, L_2, or L_1+L_2 has a resolvent of Hilbert-Schmidt type, then the operator L_1+iL_2 has a closure which has a resolvent of Hilbert-Schmidt type and its system of characteristic and adjoint elements is dense in the Hilbert space of the operator. Applications to differential operators are given. *J. L. B. Cooper* (Cardiff)

5431:

Stinespring, W. Forrest. A sufficient condition for an integral operator to have a trace. J. Reine Angew. Math. 200 (1958), 200–207.

A bounded operator T from one Hilbert space to another is a Hilbert-Schmidt operator if for some (and hence for every) orthonormal basis $\{e_\alpha\}$ of the domain, $\sum_\alpha \|Te_\alpha\|^2<\infty$. An operator is defined to be of trace class if it is the product of two Hilbert-Schmidt operators. The author proves first that if $\sum \|Te_\alpha\|<\infty$, then T is of trace class. Next, if H is a Hilbert space, $L_2(R^n, H)$ is defined as the set of all strongly measurable functions k from R^n to H that are Lebesgue square-integrable. If $k \in L_2(R^n, H)$ and if, furthermore, the support of k is contained in a cube of side 2π with edges parallel to the axes, then, for each n-tuple of integers ν, the vector

$$c_\nu=(2\pi)^{-n} \int k(x)e^{-i\nu\cdot x}dx$$

is called the νth Fourier coefficient of k. If $\sum \|c_\nu\|<\infty$, then k is said to have an absolutely convergent Fourier series. The main theorem is now as follows. Let μ be a regular measure on R^n of the form $d\mu(x)=\rho(x)dx$, where $\rho(x)$ is bounded and Lebesgue measurable, and let $k(x)$ have an absolutely convergent Fourier series as defined above. Then the operator T from $L_2(R^n, \mu)$ to H given by

$$Tf=\int f(x)k(x)d\mu(x)$$

is of trace class. Several conditions are given which imply that k has an absolutely convergent Fourier series (and hence that the corresponding T is of trace class). One of these is that

$$\sum_{j=1}^{n} \int \|\partial^r k/\partial x_j^r\|^2 dx<\infty$$

for some integer $r>\frac{1}{2}n$; for non-integer $r>\frac{1}{2}n$ this is replaced by a generalized Lipschitz condition. Finally, the case is considered in which R^n is replaced by an n-dimensional manifold of class $C^{[\frac{1}{2}n]+1}$, and the paper ends with a theorem about absolute convergence of Fourier series for compact Lie groups, which generalizes Bernstein's theorem for a real function satisfying a Lipschitz condition of order $\lambda>\frac{1}{2}$. *A. C. Zaanen* (Leiden)

5432:

Kantorovič, L. V. On integral operators. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 2(68), 3–29. (Russian)

Let D be a k-dimensional region in a real Euclidean space and D' an n-dimensional region in a real Euclidean space. Let D and D' have finite k- [n-] dimensional measure. Sometimes D and D' are to be taken as regions lying in the same Euclidean space, and then $R(P, Q)$ will denote the Euclidean distance from $P \in D$ to $Q \in D'$. For a positive real number $r\neq 1$, let $r'=r/(r-1)$, and let $1'=\infty$. The author first proves a number of theorems about integral transforms carrying various function spaces on D' into function spaces on D.

Theorem: Let $r>0$, $t>0$, $p\geqq 1$, $s\geqq 1$. Suppose that $K(P, Q)$ is a measurable function on $D\times D'$ and that

$$\left[\int_{D'} |K(P, Q)|^r dQ\right]^{1/r} \leq C_1, \quad \text{a.e. in } D,$$

$$\left[\int_{D} |K(P, Q)|^t dP\right]^{1/t} \leq C_2, \quad \text{a.e. in } D'.$$

Suppose also that $s\geqq \max(p, t)$ and that $(1-t/s)p'\leqq r$. Then the operator $V\to TV=\int_{D'} K(P, Q)V(Q)dQ$ is a bounded linear operator carrying $L^p(D')$ into $L^s(D)$, and

$$\|T\|\leq C_1^{1-t/s}C_2^{t/s}.$$

Suppose now that $0<\beta\leq 1$. Let $\operatorname{Lip}\beta$ be the class of all functions U with domain D such that

$$\|U\|_{\operatorname{Lip}\beta}=\sup\{|U(P')-U(P)|R(P, P')^{-\beta}: P, P' \in D\}<\infty.$$

(In $\operatorname{Lip}\beta$, two functions differing by a constant are identified.) Theorem: Suppose that the kernel $K(P, Q)$ has the properties that

$$\left\{\int_{D'} [|\operatorname{grad} K(P, Q)|R(P, Q)^{1-\beta}]^r dQ\right\}^{1/r} \leq E \text{ for all } P \in D$$

and

$$\left\{\int_{D'} \left[\frac{K(P, Q)}{R(P, Q)^\beta}\right]^r dQ\right\}^{1/r} \leq F \text{ for all } P\in D.$$

Then the integral operator T carries the space $L^{r'}(D')$ into $\operatorname{Lip}\beta$, and the norm of T is less than or equal to $E+(2^\beta+3^\beta)F$.

These and similar results are applied to give new proofs of and improvements on certain theorems of S. L. Sobolev [Mat. Sb. (N.S.) 2(44) (1937), 465–499; Nekotorye primeneniya funkcional'nogo analiza v matematičeskoĭ fizike, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR 14, 565]. The following is a typical example. Theorem (S. L. Sobolev's inequality): Let $U(P)$ be continuously differentiable in D, and suppose that D is convex. Suppose also that $s<np/(n-p)$. Let $\bar{U}=(1/\text{meas}(D))\int_D U(P)dP$. Then

$$\|U-\bar{U}\|_{L^s(D)} \leq A\|\operatorname{grad} U\|_{L^p(D)},$$

where A is a constant independent of U.

E. Hewitt (Seattle, Wash.)

5433:

Smolickiĭ, H. L. On summability of potentials. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 4(76), 349–356. (Russian)

Let D denote either a region of an m-dimensional Euclidean space or a smooth manifold of m dimensions

in a higher dimensional Euclidean space. In either case it is assumed that the m-dimensional measure of D is finite and different from zero. Analogous assumptions are made for the k-dimensional set D'. Let $K(P, Q)$ be a function of the points $P \in D$, and $Q \in D'$ such that for some $r>0$

$$\int_{D'} |K(P, Q)|^r dQ \leqq C_1^r$$

for almost all $P \in D$ and that for some $t>0$

$$\int_{D} |K(P, Q)|^t dP \leqq C_2^t$$

for almost all $Q \in D'$.

Let $V(Q) \in L_p$ $(1<p<\infty)$, and let

$$U(P)=\int_{D'} K(P, Q)V(Q)dQ.$$

Under these hypotheses the author proves a number of theorems. Typical results are the following: (a) If $p'=p/(p-1) \leqq r<\infty$, then $U(P)$ is essentially bounded and belongs to L_p for arbitrary p; (b) if $(1-t/p)p' \leqq r<p'$, then $U(P) \in L_q$, where $q \leqq t/(1-r/p')$; (c) if $0<r<(1-t/p)p' \leqq 1$, then $U(P) \in L_q$, where $q \leqq (t-r)/(1-r)$.

These results are generalizations of certain ones obtained by L. V. Kantorovič [see the article reviewed above]. *H. P. Thielman* (Ames, Iowa.)

5434:

Putnam, C. R. Commutators and normal operators. Portugal. Math. 17 (1958), 59–62.

If A and B are bounded operators on a Hilbert space and $C=AB-BA$, let W_C be the closure of the set of numbers (Cx, x) for $\|x\|=1$ and let W_C' be W_C if W_C is a singleton, or the open interval if W_C is a closed interval, or the interior of W_C otherwise. In an earlier paper [Proc. Amer. Math. Soc. 7 (1956), 1026–1030; MR 18, 495] the author showed that 0 is in W_C' if A is self-adjoint with a spectrum of measure zero. This result does not extend to normal operators; however, if $A=H+iJ$, H and J self-adjoint, J has a pure point spectrum and the spectrum of H has measure zero, then 0 is in W_C'. A more general criterion is also given.

D. C. Kleinecke (Livermore, Calif.)

5435:

Yosida, Kôsaku. On the differentiability of semi-groups of linear operators. Proc. Japan Acad. 34 (1958), 337–340.

"Let $R(\lambda, A)$ be the resolvent $(\lambda I-A)^{-1}$ of the infinitesimal generator A of a semi-group T_t, $t \geqq 0$, of bounded linear operators on a complex Banach space X, strongly continuous in t, such that $T_0=I$ and $\|T_t\| \leqq 1$ (for all $t \geqq 0$). As suggested by the researches of E. Hille and R. S. Phillips [Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., New York, 1957; MR 19, 664], the author proves the theorem: T_tX belongs, for all $t>0$, to the domain $D(A)$ of A in such a way that $\limsup_{t\downarrow 0} t\cdot\|AT_t\|<\infty$ if and only if $\limsup_{|\tau|\uparrow\infty} |\tau|\cdot\|R(1+(-1)^{1/2}\tau, A)\|<\infty$. Moreover, the semi-group T_t of such class can be extended to T_λ which is analytic in a sector $|\lambda-t|<t(eC)^{-1}$ $(C=\sup_{t>0}\|\exp(-t/2)AT_t\|)$ of the complex λ-plane. Theorem 2 of the present paper must be corrected as: $\limsup_{|\tau|\uparrow\infty}\|R(1+(-1)^{\frac{1}{2}}\tau, A)\|\log|\tau|=0$ implies that $T_tX \in D(A)$ for any $t>0$; and the converse is true if, in addition, we have $\limsup_{t\downarrow 0} t\cdot\log\|AT_t\|=0$."

From a summary provided by the author

5436:

Katznelson, Yitzhak. Algèbres caractérisées par les fonctions qui opèrent sur elles. C. R. Acad. Sci. Paris 247 (1958), 903–905.

Let B be a commutative semi-simple Banach algebra, represented as an algebra of continuous functions on its maximal ideal space M. The principal result of this note is the following theorem: If $F(f)$ belongs to B for every f in B and for every continuous function F defined in the complex plane, then M is compact and B is the algebra of all continuous functions on M. The idea of the proof is to study the action of the function $F(x)=x^{\frac{1}{2}}$ on the non-negative functions in B; the author deduces the existence of a constant K such that $\|f\|^2 \leqq K\|f^2\|$, and this shows that the given norm in B is equivalent to the uniform norm.

W. Rudin (New Haven, Conn.)

5437:

Sakai, Shôichirô. On topological properties of W^*-algebras. Proc. Japan Acad. 33 (1957), 439–444.

(1) A set V in the dual $\bar{A}$ of a C^* algebra A is called invariant if it is invariant under the adjoint operations to left and right multiplication in A. Then it is shown that any closed invariant subspace V of $\bar{A}$ is algebraically spanned by positive functionals in V. As a consequence: if A is a W^*-algebra and A_* the space of all σ-weakly continuous linear functionals on A, then A_* is $\sigma(A_*, A)$-sequentially complete.

(2) It had been conjectured, in analogy with the commutative case, that in the Banach space T of all trace-class operators in a separable Hilbert space, in the l_1-norm, norm-convergence is equivalent to weak convergence. A counter-example to this is given.

(3) Certain topological differences between purely infinite and semifinite W^*-algebras are shown, from which it is deduced that the direct product of two W^*-algebras is purely infinite if either factor is such.

J. Feldman (Berkeley, Calif.)

5438:

Tamme, È. È. On implicit operators. Dokl. Akad. Nauk SSSR 120 (1958), 259–261. (Russian)

It is assumed that X, Y, and Z are Banach spaces and that $F:X\times Y\to Z$ is analytic at a point (x_0, y_0) where F vanishes and the continuous operator $(F_x)^{-1}$ exists. Then it is known [T. H. Hildebrandt and L. M. Graves, Trans. Amer. Math. Soc. 29 (1927), 127–153] that the unique solution $x=\Phi(y)$ with $x_0=\Phi(y_0)$ of the equation $F(x, y)=0$ exists in a neighborhood of y_0 and is analytic. By dominating the expansion of F around (x_0, y_0) by a scalar function, one can obtain information on the radius of convergence of Φ as well as the rapidity of convergence of the partial sums of Φ. The author suggests an application to the perturbation of an eigenvalue problem $Ax=\lambda Bx$, similar to the method used by P. C. Rosenbloom [Arch. Math. 6 (1955), 89–101; MR 16, 832]. Details are not given.

R. G. Bartle (Urbana, Ill.)

5439:

Collatz, L. Näherungsverfahren höherer Ordnung für Gleichungen in Banach-Räumen. Arch. Rational Mech. Anal. 2 (1958), 66–75.

Iteration procedures for the location of a root of $Tf=0$ in a Banach space are considered. It is assumed that T is defined on a closed convex region and possesses Fréchet derivatives $T', T'', \cdots$ of various orders. Iterations of the general form

$$f_{n+1}=f_n+\Phi(T'(f_n)^{-1}, T(f_n), T'(f_n), T''(f_n), \cdots)$$

are considered and it is assumed that there are monotone

non-decreasing functions G, H and a constant $k \geqq 1$ such that

$$\|T(f_{n+1})\| \leqq G_n \|T(f_n)\|^k,$$
$$\|f_{n+1}-f_n\| \leqq H_n \|T(f_n)\|,$$

where G_n and H_n denote G and H evaluated at the point $(\|T'(f_n)^{-1}\|, \|Tf_n\|)$, respectively. If $\|T''\|$ is bounded and certain other bounds are satisfied, then it is seen that the iteration is possible and converges to a solution u of $Tf=0$ with the estimate

$$\|u-f_n\| \leqq \text{const}(\beta^{1+k+\cdots+k^{n-1}})/(1-\beta^{k^n})$$

for some β with $0<\beta<1$. Newton's method and some of its higher-order modifications are examples of this type of iteration. An application is given to a special matrix eigenvalue problem of the form $Ax=\lambda Bx$.

R. G. Bartle (Urbana, Ill.)

5440:

Altman, M. On the approximate solutions of non-linear functional equations in Banach spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 19–24.

This note gives an iterative method for the location of a solution of $F(x)=0$, where F is a real-valued function defined near a point x_0 of a Banach space. The process is:

$$x_{n+1}=x_n-\frac{F(x_n)}{F'(x_n)(y_n)}y_n \quad (n=0, 1, 2, \cdots),$$

where y_n is to be chosen such that $\|y_n\|=1$ and $F'(x_n)(y_n) = \|F'(x_n)\|$. Among other things it is assumed that F has two Fréchet derivatives and that there is a real-valued function Q defined on a real interval (z_0, z') and possessing a root in this interval such that $|F(x_0)| \leqq Q(z_0)$ and $\|F''(x)\| \leqq Q''(z)$ for $\|x-x_0\| \leqq z-z_0$.

R. G. Bartle (Urbana, Ill.)

CALCULUS OF VARIATIONS

See also 5356, 5357, 5361.

5441:

Эльсгольц, Л. Э. Вариационное исчисление. [Èl'sgol'c, L. È. The calculus of variations.] 2nd ed., corrected. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1958. 163 pp. 5.65 rubles.

The first edition (1952), which is essentially the same as the present one, was reviewed in MR **14**, 482.

5442:

Cinquini, Silvio. Sopra le estremaloidi di una classe di problemi variazionali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 116–120.

The problem is to minimize $\int F(x(s), y(s), x'(s), y'(s), \theta'(s))ds$ where s is arclength, $\cos\theta=x'$, $\sin\theta=y'$, and F is positively homogeneous of degree one in x' and y'. An "ordinary curve" is a rectifiable curve meeting specified boundary conditions for which the components x, y and their derivatives x', y' are absolutely continuous and for which the integral exists. In an earlier paper [same Rend. **23** (1957), 22–28; MR **20** #3470] the author determined the Euler equations for this problem in differential form. An ordinary curve for which θ' is finite and continuous and whose components satisfy these equations is an "extremal of order 2". An "extremaloid of order 2" is an ordinary curve such that (i) the integrands appearing in the integral form of the Euler equations are integrable and (ii) the components of the curve satisfy the Euler equations in integral form. The paper gives conditions under which an ordinary curve which minimizes the integral is an extremaloid, and conditions under which an extremaloid is an extremal.

E. Silverman (Lafayette, Ind.)

5443:

Ladyženskaya, O. A. Differential properties of generalized solutions of some n-dimensional variational problems. Dokl. Akad. Nauk SSSR **120** (1958), 956–959. (Russian)

The functional $F(u)=(\Omega)\int F(u_1, \cdots, u_n)dx$ is studied, in which $dx=dx_1\cdots dx_n$, $u_i=\partial u/\partial x_i$, and Ω is a closed bounded region in E_n. If $\|u\|$ denotes the Euclidean norm, $F_i=\partial F/\partial x_i$, $F_{ij}=\partial^2 F/\partial x_i \partial x_j$, then F is assumed to be of class C^2 with

$$m_1\|u\|^p \leqq F \leqq M_1(\|u\|^p+1),$$
$$m_2(\|u\|^{p-2}+1) \leqq F_{ij} \leqq M_2(\|u\|^{p-2}+1),$$
$$|F_i| \leqq M_3(\|u\|^{p-1}+1),\ m_1>0,\ m_2>0,\ p \geqq 2.$$

The admissible boundary values on the boundary S of Ω are supposed to be given by functions $\varphi(x)$, $x \in \Omega$, defined on all of Ω and forming a space W_p^k. This space consists of all $\varphi(x)$, $x \in \Omega$, summable on Ω with their power p, together with all the generalized partial derivatives $D^{(s)}u$ up to the order k. The norm in W_p^k is defined by

$$\|u\|_W=\{(\Omega)\int \sum |D^{(s)}u|^p dx\}^{1/p},$$

where $\sum$ ranges over all derivatives $D^{(s)}u$ of the orders $|s|=0, 1, \cdots, k$, $D^{(0)}u=u$. A space $\mathring{W}_p^k$ is obtained by completion in the norm W_p^k of the set of all continuously differentiable φ which are equal to zero on S. Then the following variational problem (which is easier than the usual n-dimensional problem) is considered: find the minimum of $I(u)$ in the class of all $u(x)$, $x \in \Omega$, with $u(x)-\varphi(x) \in \mathring{W}_p^1$. A function of this class realizing the minimum is said to be a generalized solution of the variational problem. Under a "three point condition", which cannot be given here, it is proved that a generalized solution is continuous and that it satisfies a Lipschitz condition and the Euler equation almost everywhere in Ω. [Pertinent references: V. I. Kazimirov, Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 125–129; MR **18**, 217; G. I. Silova, Dokl. Akad. Nauk SSSR **102** (1955), 699–702; MR **17**, 46; and the well known work of Tonelli; Morrey, Univ. Calif. Publ. Math. (N.S.) **1** (1943), 1–130; Contributions to the theory of differential equations, pp. 101–159, Princeton Univ. Press, 1954; MR **6**, 180; **16**, 827; Nirenberg, ibid. pp. 95–100; MR **16**, 592; Stampacchia, Ricerche Mat. **1** (1952), 200–226; MR **15**, 328; and others.]

L. Cesari (Baltimore, Md.)

GEOMETRIES, EUCLIDEAN AND OTHER

See also 5139, 5711.

5444:

Smogorževskiĭ, A. S. On polygons. Izv. Kiev. Politehn. Inst. **16** (1954), 184–199. (Russian)

Theorems are proved on the basis of Hilbert's axioms of combination and order, without recourse to those of congruence, parallelism and continuity.

5445:
Karzel, Helmut. Spiegelungsgeometrien mit echtem Zentrum. Arch. Math. 9 (1958), 140–146.

In the group-theoretic axiomatization of plane geometry the author omits the axiom that the centre be trivial and is thus led to new geometries. Let Γ be a group generated by a set of involutions Ω. (The elements of Ω are to be interpreted as reflections in lines which they represent). That α is an involution is expressed by α inv. The axioms are the following. (I) If $a\neq b$ and abx_i inv, $i=1, 2, 3$, then $x_1x_2x_3\in\Omega$. — For $a\neq b$ let $p_{ab}=\{x:abx$ inv$\}$. Then p_{ab} is called a pencil. (II) For a given line a there are at least two pencils p_1, p_2 with $a\in p_i$. — A pencil p_{ab} is proper when $p_{ab}\cap p_{cd}\neq 0$ for any pencil p_{cd}. (III) If ab inv, then p_{ab} is proper. We assume that the center Z of Γ does not consist of 1 alone. There is exactly one commutative Γ satisfying I, II, III, namely the so-called "Vierergruppe of F. Klein"; Ω contains then all involutions. The geometry is non-degenerate if there are a, b, c not in Z with $(abc)^2\neq 1$. Every non-degenerate geometry is a projective plane over a commutative field of characteristic 2. The degenerate geometries have a very simple structure; they consist of one pencil of lines not in Z and a single line in Z.

H. Busemann (Cambridge, Mass.)

5446:
Lenz, Hanfried. Über ebene Drehungen. Arch. Math. 8 (1957), 477–480.

In the ordinary affine plane consider a group Γ of affinities with the properties: Γ is transitive on the lines through the origin O; by affinities in Γ a point $P\neq O$ can be carried into only a finite number of points of the line PO. It is proved that Γ is either the group of rotations about O, or this group augmented by the reflections in lines through O, or, finally, the elements of Γ are representable in the form $e^{f(\varphi)}\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$, where f is a solution of the functional equation $f(\varphi+\psi)=f(\varphi)+f(\psi)$ with $f(\pi)=0$. The latter group had been used previously by Pickert as a counterexample [Math. Ann. **120** (1949), 492–501; MR **10**, 571]. *H. Busemann* (Cambridge, Mass.)

5447:
Lenz, Hanfried. Über räumliche Drehungen. Math. Ann. 135 (1958), 244–250.

Sharpening results of G. Pickert [cf. #5446] and R. Baer [Trans. Amer. Math. Soc. **68** (1950), 439–460; MR **12**, 9], the author proves the following theorem. Let $\mathfrak{G}$ be a group of homogeneous affine transformations of the three-dimensional real space. For $\mathfrak{G}$ to be an orthogonal group (belonging to a positive definite quadratic form which is uniquely determined except for a constant factor), it is necessary and sufficient that (I) there are no lines or planes invariant under all transformations of $\mathfrak{G}$, and (II) the real eigenvalues of the transformations of $\mathfrak{G}$ are uniformly bounded (hence equal to ± 1). In n-dimensional space ($n>3$) conditions somewhat stronger than (I), (II) are shown to be sufficient [for an improvement of these results see H. Lenz, Eine Kennzeichnung der euklidischen Drehgruppen, to appear in Arch. Math.]. The case $n=2$, in which the above theorem is false, was treated in an earlier paper [#5446].

F. A. Behrend (Victoria)

5448:
Bottema, O. The theorem of Pompeĭu. Nieuw Tijdschr. Wisk. 44 (1956/57), 183–184. (Dutch)

G. R. Veldkamp [same Tijdschr. **44** (1956/57), 1–4; MR **18**, 328] generalized the Pompeĭu theorem for an equilateral triangle ABC and a point P outside of plane ABC. Here another proof of this generalization is given, using the formula in which the volume I of tetrahedron $PABC$ is expressed in terms of its six edges. If ABC is equilateral with side a and area O, then $a^2O^2\geqq 12I^2$; the equality holds when P lies on the sphere through A, B, C whose center is at the center of gravity of ABC. [Cf. J. Hadamard, "Leçons de géométrie élémentaire" vol. II, Libr. Armand Colin, Paris, 1898; exerc. 873, 874 on p. 315.]

S. R. Struik (Cambridge, Mass.)

5449:
Boomstra, W. The theorem of Pompeĭu again. Nieuw Tijdschr. Wisk. 44 (1956/57), 285–288. (Dutch)

A triangle $A_1A_2A_3$ with the sides a_i, $(i=1, 2, 3)$ has, with respect to a point A_4, the pedal triangle PQR. If r_i are the distances from A_4 to the vertices of the triangle then $QR/a_1r_1=RP/a_2r_2=PQ/a_3r_3$. This known theorem gives the Pompeĭu theorem as special case for $a_1=a_2=a_3$. A further generalization allows A_4 to be outside of the plane of $a_{1,2,3}$. Thus: the three products of opposite edges of a tetrahedron are numbers which can always be considered as the sides of a triangle.

S. R. Struik (Cambridge, Mass.)

5450:
Wheeler, R. F. The flexagon family. Math. Gaz. 42 (1958), 1–6.

A flexagon is a strip (not necessarily straight) of $3t$ equilateral triangles ($t\geqq 3$) suitably folded into a regular hexagon, the ends being joined. (The author requires the strip to be planar, but this restriction is unnecessary.) By a characteristic folding operation, the faces of the triangles can be displayed in sets of 6; faces in the same set may be supposed to be coloured alike. It is assumed that all t colours can be displayed in this way, "flaps" being excluded, and (tacitly) that the configuration has triad symmetry. The author introduces a diagrammatic representation ("map") of the structure of a flexagon, and uses it to give a systematic method for colouring and folding the simplest class of flexagons, formed from straight strips with $t=3\cdot 2^k$. He gives examples of other flexagons, and briefly considers some generalisations. [The systematic method could be extended to the general flexagon; this was done in 1940 by R. Feynman, B. J. Tuckerman, J. W. Tukey and the reviewer, but never published. Further references: C. O. Oakley and R. J. Wisner, Amer. Math. Monthly **64** (1957), 143–154; MR **19**, 240; M. Gardner, Scientific American **195** (1956), no. 6, 162–166; **198** (1958), no. 5, 122–126.]

A. H. Stone (Manchester)

5451:
Clement, Paul A. The concurrency of perpendiculars. Amer. Math. Monthly 65 (1958), 601–605.

If the six segments which the feet of the three perpendiculars from a point to the sides of a triangle determine on those sides are grouped into two sets of non-consecutive segments, the sum of the squares of the segments of one set is equal to the analogous sum of the second set, and conversely.

The author proves the above proposition, derives a modified, more serviceable form of the formula involved, and uses it to prove the classical property that two points isogonal for a triangle have a common pedal circle [see, for instance, N. A. Court, College geometry, 2nd. ed., Barnes and Noble, New York, 1952; MR **14**, 307; p. 271].

N. A. Court (Norman, Okla.)

5452:

Benz, Walter. (8_3, 6_4)-Konfigurationen in Laguerre-, Möbius- und weiteren Geometrien. Math. Z. **70** (1958), 283–296.

The author considers, in the conformal plane, two configurations of 6 circles passing by threes through 8 points. Both configurations can be derived from a cube by stereographic projection from a point of general position on its circumsphere. In one case the 6 circles come from the circumcircles of the 6 faces; in the other case they come from the circumcircles of the rectangles formed by pairs of opposite edges. Each configuration involves a theorem to the effect that if 5 of the 6 indicated sets of 4 points are concyclic, the remaining set likewise consists of 4 concyclic points. In the former case this is Miquel's theorem [A. Miquel, J. Math. Pures Appl. (1) **10** (1845), 348–349]. When the points of the plane are represented in the usual way by complex numbers, the circles correspond to chains whose definition is associated with the real subfield of the complex field. The author considers the effect of replacing the complex field and its real subfield by a general commutative ring and a subfield satisfying certain conditions. These conditions can be less restrictive in the case of Miquel's theorem than in the case of the other theorem.

H. S. M. Coxeter (Toronto, Ont.)

5453:

Havel, Václav. Fundamentalsätze der mehrdimensionalen Zentralaxonometrie. II, III. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **8** (1958), 103–114. (Czech. Russian and German summaries)

"Im Teil II, § 1 führt der Verfasser einige Ergänzungen zu den beiden Fundamentalsätzen an [siehe dieselbe Časopis **7** (1957), 94–107; MR **20**#4219; S. 106–107]. Im § 2 werden die Bedingungen, unter denen zwei Simplizes sich bis auf die Ähnlichkeit in einer perspektiven Affinität entsprechen, untersucht. Im § 3 wird ein Resultat von H. Naumann [siehe Math. Z. **67** (1957), 75–82; MR **18**, 921; Satz 1. 3, S. 78] synthetisch verallgemeinert. Es handelt sich im wesentlichen um eine Verallgemeinerung des klassischen Satzes von Pohlke und Schwarz. Im Teil III ist die Projektion der desarguesschen Konfiguration und der sog. (n, m)-Konfiguration in die Gerade [vergleiche F. Hohenberg, Monatsh. Math. **61** (1957), 54–66; MR **18**, 921; S. 55] untersucht." *F. A. Behrend* (Melbourne)

5454:

Medek, Václav. Zyklographische Abbildung in der Ebene. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **8** (1958), 73–80. (Slovak. Russian and German summaries)

The points A of a projective plane are mapped onto point pairs ${}^1A$, ${}^2A$ of a line p by projection from two centres ${}^1O$, ${}^2O$. The author studies the projectivity ${}^1A \to {}^2A$ produced when A describes either a line or a conic through ${}^1O$, ${}^2O$, and classifies the pencils of such projectivities corresponding to pencils of lines or conics.

F. A. Behrend (Melbourne)

5455:

Sasayama, Hiroyoshi. General coordinate geometries. I. J. Spatial Math. Sasayama Res. Room **1** (1958), 159–186.

An expository paper on Hilbert space. The author follows Euclidean geometric ideas as closely as possible.

5456:

Sasayama, Hiroyoshi. On the quasi non-euclidean geometry of the absolute of arbitrary real order. Rend. Circ. Mat. Palermo (2) **6** (1957), 334–342.

"The purpose of the present article is to introduce the general quasi non-euclidean space of quasi orthogonality with respect to a hypersurface of arbitrary real order ρ as absolute, which gives new theories even in 2- and 3-dimensional cases for a non-integral real order ρ, e.g., the plane geometry with an astroid as absolute." (From the author's summary) *L. A. Santaló* (Buenos Aires)

5457:

Bukreev, B. Ya. Lobačevskian geometry. Kiïv. Derž. Univ. Nauk Zap. **16** (1957), no. 2=Kiev. Gos. Univ. Mat. Sb. **9** (1957), 5–9. (Russian)

5458:

Straszewicz, S. On the basic theorems of Lobaczevski's trigonometry. Prace Mat. **2** (1958), 269–286. (Polish. Russian and English summaries)

The paper opens with an introduction giving historical and bibliographical data regarding the topic considered. The author admits the same axioms as did L. Gerard and arrives at the fundamental propositions of hyperbolic trigonometry by the use of concise proofs based largely on elementary synthetic geometry.

N. A. Court (Norman, Okla.)

CONVEX SETS AND DISTANCE GEOMETRIES

See also 5157, 5186, 5376, 5499, 5500, 5501.

5459:

Hirschfeld, R. A. On a minimax theorem of K. Fan. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 470–474.

In a paper by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 42–47; MR **14**, 1109] a real-valued function f defined on the product set $X \times Y$ of two sets X, Y is said to be convex on X if, for any two elements x_1, x_2 of X and any $\alpha \geq 0$, $\beta \geq 0$ with $\alpha+\beta=1$, there exists an $x_0 \in X$ such that $f(x_0, y) \leq \alpha f(x_1, y) + \beta f(x_2, y)$ for all $y \in Y$. The concavity of f on Y is defined similarly. With this definition the reviewer obtained the following minimax theorem, which does not presuppose the structure of vector spaces, and generalizes a result of H. Kneser [C. R. Acad. Sci. Paris **234** (1952), 2418–2420; MR **14**, 301]. Let X be a compact Hausdorff space and Y an arbitrary set (not topologized). Let f be a real-valued function on $X \times Y$ such that, for every fixed $y \in Y$, $f(x, y)$ is lower semi-continuous on X. If f is convex on X and concave on Y, then

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y).$$

The present paper gives a new proof of this theorem. The author's proof, quite different from the reviewer's, uses an argument of H. Nikaidô [J. Math. Soc. Japan **5** (1953), 86–94; MR **15**, 324] and reduces the problem to the separation theorem for convex sets in a Euclidean space. As noted by the author, still another proof based on a covering theorem of B. Knaster, C. Kuratowski and S. Mazurkiewicz [Fund. Math. **14** (1929), 132–137] is contained in a recent paper by M. Sion [Pacific J. Math. **8** (1958), 171–176; MR **20**#3506]. *Ky Fan* (Notre Dame, Ind.)

5460:

Hadwiger, H. Ueber Eibereiche mit gemeinsamer Treffgeraden. Portugal. Math. **16** (1957), 23–29.

Der folgende Satz ist bewiesen: Ist M eine endliche oder

abzählbare unendliche Menge in der Ebene disjunkt liegender Eibereiche, so existiert dann und nur dann eine Gerade, die alle Eibereiche von M trifft, wenn sich in der Menge M eine Ordnungsbeziehung derart festlegen lässt, dass je drei Eibereiche von M von einer geeigneten Geraden in der Reihenfolge der in M festgelegten Ordnung getroffen werden. *L. A. Santaló* (Buenos Aires)

5461:

Kijne, D. On collinearities of twodimensional convex bodies. Nieuw Arch. Wisk. (3) **5** (1957), 81–83.

Plane convex sets G_σ are called collinear if they have a common secant. A family $F^{(n)}$ of n plane convex sets G_σ is said to possess the property B_i for some $i \leqq n$ if each of its $\binom{n}{i}$ subfamilies $F_k^{(i)}$ $(k=1, 2, \cdots, \binom{n}{i})$ consists of collinear G_σ. The author gives a method for the construction of families $F^{(n)}$ of plane convex sets of arbitrary shape (e.g., circles) with the property B_{n-1} but not with the property B_n. *L. A. Santaló* (Buenos Aires)

5462:

Abel, William R.; and Blumenthal, Leonard M. Distance geometry of metric arcs. Amer. Math. Monthly **64** (1957), no. 8, part II, 1–10.

The main results of the paper are the following. (1) If a metric arc $A(a, b)$ has a finite Menger curvature at each point, then a positive integer N exists such that, for $n > N$, (i) every n-lattice L_n of $A(a, b)$ is a homogeneous $\lambda(n)$-chain, where $\lambda(n)$ is the distance of two consecutive points of L_n, and (ii) the arc $A(a, b)$ contains exactly one n-lattice. (2) The length of any metric arc (rectifiable or not) is the limit of the lengths of inscribed n-lattices. (3) Every metric ptolemaic arc that is a geodesic is a metric segment. For the nomenclature see the book of Blumenthal, Theory and applications of Distance Geometry, Clarendon Press, Oxford, 1953 [MR **14**, 1009].
L. A. Santaló (Buenos Aires)

GENERAL TOPOLOGY, POINT SET THEORY

See also 5146, 5151, 5247, 5268, 5459, 5501, 5512.

5463:

Roux, Delfina. Sull'isolamento rispetto a sistemi di punti. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957), 107–116.

The object of this paper is to find the abstract content of the Boutroux-Cartan lemma on the minimum modulus of a polynomial. In a Euclidean space consider a set Z of points z (repetitions allowed), and let $N(x, \rho)$ be the number of points z in a ball of center x and radius ρ. Let $\{\delta_h\}$ be an increasing sequence of positive numbers, and let $h(\rho)=h$ for $\delta_{h-1} < \rho \leqq \delta_h$. If $N(y, \rho_0) \geqq h(\rho_0)$, the ball $|x-y| \leqq \rho_0$ is called a ball of δ-density. Then y is called δ-isolated if it is not the center of any ball of δ-density. The fundamental theorem is as follows. If Z contains n points, it can be partitioned into $p \leqq n$ disjoint sets Z_k containing, respectively, ν_k points z, so that if c_k is the center and ρ_k is the radius of the smallest ball containing z_k, we have $\rho_k \leqq \delta_{\nu_k}$, and the set of points z that are not δ-isolated is contained in the union of the p balls $|x-c_k| \leqq \rho_k + \delta_{\nu_k}$. The Boutroux-Cartan lemma, in a slightly sharpened form, follows readily from this.
R. P. Boas, Jr. (Evanston, Ill.)

5464:

Bruns, Günter; und Schmidt, Jürgen. Die punktalen Typen topologischer Räume. Math. Japon. **4** (1957), 133–177.

A filter $\mathcal{F}$ in a set E is a family of subsets of E which contains finite intersections and all supersets of its members. Two filters are isomorphic if there is a one-to-one mapping of the sets in which they lie which induces a one-to-one correspondence between the members. If $F \in \mathcal{F}$, the full basis of $\mathcal{F}$ in F is the set of all members of $\mathcal{F}$ which are subsets of F. Two filters are similar if they have isomorphic full bases; the type of a filter is the set of all similar filters. The punctal type of a topological space E in a point x is the filter type of the neighborhood system of x in E.

This paper investigates various decompositions of filters and various invariants of the filter type, many of them cardinal numbers. Since cofinal equivalence of the directed systems of members of two filters is a consequence of similarity of the two filters, the cofinal type determined by $\mathcal{F}$ is also an invariant of the filter type. The question is raised as to what other invariants might be needed to determine the filter type from the cofinal type. *M. M. Day* (Urbana, Ill.)

5465:

Bruns, Günter. Die Struktur der unverzweigten punktalen Raumtypen. Math. Japon. **4** (1957), 123–132.

[See the above review for definitions.] This paper shows that three cardinal number invariants of the paper reviewed above suffice to determine the isomorphism type of an unbranched filter (that is, one whose cofinal type is well-ordered). *M. M. Day* (Urbana, Ill.)

5466:

Kerstan, Johannes. Eine neue mengentheoretische Charakterisierung der vollständig regulären Räume. Rev. Math. Pures Appl. **1** (1956), no. 3, 33–34.

This is an announcement of a later paper. See Math. Nachr. **17** (1958), 27–46 [MR **20** #1968] for a statement of the results. *H. H. Corson* (Seattle, Wash.)

5467:

Kasahara, Shouro. A note on some topological spaces. Proc. Japan Acad. **33** (1957), 453–454.

Theorem 1 lists six conditions on families of open subsets of a Hausdorff space E, all obviously equivalent to finiteness of E (for example, every family of pairwise disjoint open subsets is finite). Theorem 2: A topological space E is feebly compact if and only if every locally finite family of open subsets of E is finite. [See Mardešić and Papić, Hrvatsko Prirod. Društvo. Glasnik Mat. Fiz. Astr. Ser. II. **10** (1955), 225–232; MR **18**, 224; Iséki and Kasahara, Proc. Japan Acad. **33** (1957), 100–102; MR **19**, 668; Kasahara, ibid. **33** (1957), 255–259; MR **19**, 668.]
E. Hewitt (Seattle, Wash.)

5468:

Atsuji, Masahiko. Uniform continuity of continuous functions of metric spaces. Pacific J. Math. **8** (1958), 11–16; erratum, 941.

Theorem 2: The following condition on a metric space S is necessary and sufficient in order that every uniformly continuous function on S be bounded: Given $\varepsilon > 0$, there exist an integer m and a finite set $P \subset S$ such that for each $x \in S$ there exist points $x_i \in S$, $i=1, \cdots, m$, and $p \in P$, satisfying $d(p, x_1) < \varepsilon$, $d(x_i, x_{i+1}) < \varepsilon$, and $d(x_m, x) < \varepsilon$.

Theorem 3: If a connected metric space S admits a uniformly continuous unbounded function, then the set of all uniformly continuous functions on S is not closed under pointwise multiplication.

M. Jerison (Princeton, N.J.)

5469:

Hanai, Sitiro. On open and closed mappings. Mem. Osaka Univ. Lib. Arts Ed. Ser. B. **4** (1955), 51–55.

Let S and E be T_1-spaces. By a mapping $f(S)=E$ is meant a single-valued continuous transformation. A mapping is open [closed] if and only if the image of every open [closed] set in S is an open [closed] set in E. The author considers a directed system of sets $\{X_\lambda | \lambda \in \Lambda\}$ in S. The limit superior and limit inferior of this system are defined in the natural way, and the system is said to be convergent if and only if $\lim\sup X_\lambda = \lim\inf X_\lambda$. We write $\lim X_\lambda = \lim\sup X_\lambda = \lim\inf X_\lambda$ if the system is convergent. Clearly, both the limit inferior and the limit superior of a system of sets are themselves closed sets.

The author shows that if E satisfies the first countability axiom and if $f(S)=E$ is an open mapping, then for any sequence $\{X_n\}$ of inverse sets of S which converges to a set X, we must have $\lim f(X_n)=f(X)$.

The remaining theorems of the paper all place the additional assumption on the mapping $f(S)=E$ that $f^{-1}(y)$ is a compact set for every point y in E. The first two of the these theorems are the following. (A) Let E be a T_2-space and let f be closed. Then for any directed system of points $\{y_\lambda\}$ of E such that $y_\lambda \to y$, $\lim\sup f^{-1}(y_\lambda)$ is a non-empty subset of $f^{-1}(y)$. (B) Let E be a T_2-space. Then f is closed if and only if f satisfies the conclusion of (A) and the decomposition $\{f^{-1}(y)\}$ generated by f is upper semi-continuous.

A T_2-space [regular space] S is said to be absolutely H-closed [absolutely r-closed] if and only if every homeomorphic image of S which is a subset of a T_2-space [regular space] E is a closed subset of E. Absolute H-closedness is invariant under a mapping, and the author states that he can easily prove that absolute r-closedness is also invariant under a mapping. Retaining the assumption on the mapping $f(S)=E$ mentioned in the previous paragraph, he establishes the following additional theorems. (C) Let S and E be T_2-spaces and f a mapping which is both open and closed. Then absolute H-closedness is invariant under the inverse transformation f^{-1}. If the condition that f be open is dropped in this theorem, then S is not necessarily absolutely H-closed. (E) Let S and E be regular spaces and f a mapping which is both open and closed. Then absolute r-closedness is invariant under the inverse transformation f^{-1}.

D. W. Hall (Endicott, N.Y.)

5470:

Seibert, Peter. On a problem of Mazurkiewicz concerning the boundary of a covering surface. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 50–54.

Given an interior map s of a 2-dimensional manifold F into S^2, a metric on F is introduced by $\rho(p,q)=\inf_C \operatorname{diam} s(C)$, "diam" being diameter in the sense of the spherical metric on S^2 and C being a Jordan arc on F with endpoints p and q. With F^* denoting the completion of F in the sense of ρ, the boundary Z of F (relative to F^*) is introduced. The map s admits a continuous extension to F^*; this extension will also be denoted by s. The principal results of the paper are the following. (1) If F is simply-connected and Z is locally compact, then $\dim Z \leq 1$. (2) There exists $s: F \to S^2$ for some F such that s maps Z isometrically onto S^2. A corresponding result holds with s mapping Z isometrically onto a closed subset A of S^2. A problem of Mazurkiewicz [Fund. Math. **17** (1931), 26–29] is thereby settled. The proof of the second cited result is by construction. Use is made of results of the author which are to appear in Math. Nachr.

M. H. Heins (Urbana, Ill.)

ALGEBRAIC TOPOLOGY

See also 5299.

5471:

Wu, Wen-tsün. On the realization of complexes in euclidean spaces. I. Sci. Sinica **7** (1958), 251–297.

Translation of a Chinese original [Acta Math. Sinica **5** (1955), 505–552; MR **17**, 883].

5472:

Mardešić, S. Equivalence of singular and Čech homology for ANR-s. Application to unicoherence. Fund. Math. **46** (1958), 29–45.

In § 1 the author gives a direct proof of the known result: In a metric ANR for metric spaces, singular homology coincides with Čech homology based on all coverings. {Reviewer's remark: This equivalence is known to hold more generally in any space dominated by a polytope: J. Dugundji, Portugal. Math. **14** (1955), 39–41 [MR **19**, 974].} § 2: Main theorem: Let X be compact metric, $\dim X=k$. If $k\leq m-2$, the function space $(S^m)^X$ is unicoherent; if $k=m-1$, $(S^m)^X$ is unicoherent if and only if $\operatorname{Hom}[H^{m-1}(x), Z]=0$ (integral Čech cohomology, Z=integers). This contains Ganea's result [C. R. Acad. Sci. Paris **242** (1956), 725–728; MR **17**, 881] that $(S^2)^{S^1}$ is not unicoherent. The section also contains a detailed proof showing that Eilenberg's theorem: "$f: Y\to S^1$ is nullhomotopic if and only if there is a $\varphi: Y\to E^1$ with $f(y)=e^{i\varphi(y)}$", originally established for metric Y, is in fact valid for arbitrary Y. {This generalization is not new; it can be found (with different proof) in Kuratowski, "Topologie II", Monografie Mat. vol. 21, Warszawa-Wrocław, 1950 [MR **12**, 517], p. 326.}

J. Dugundji (Los Angeles, Calif.)

5473:

***Thomas, Emery. The generalized Pontrjagin cohomology operations.** Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 155–158. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

This note is a summary of an address given in Mexico City in August, 1956. In it the author summarizes (without proofs) the principal properties of the generalized Pontrjagin cohomology operations. Since then the author has published a complete account (including proofs) of these new cohomology operations [#5474 below].

There is one point discussed in this short note which is not covered in the author's longer memoir. There is an example of two simply connected 6-dimensional cell complexes which are not of the same homotopy type. To prove this, the author uses his generalized Pontrjagin cube. None of the classical cohomology operations suffice to distinguish these two complexes. The author also indicates how one could construct examples of complexes

whose homotopy type could only be distinguished by generalized Pontrjagin pth power for any given prime p.
W. S. Massey (Providence, R.I.)

5474:

Thomas, Emery. The generalized Pontrjagin cohomology operations and rings with divided powers. Mem. Amer. Math. Soc. no. 27 (1957), 82 pp.

L'Auteur expose en détail sa théorie des "opérations de Pontrjagin", qui généralisent le "carré de Pontrjagin". Soit $\Gamma(\mathscr{R})$ la catégorie des anneaux gradués A à puissances divisées; on a donc des applications γ_r $(r=0, 1, 2, \cdots)$ qui, pour tout entier $k\geqq 1$, envoient A_{2k} dans A_{2rk}, avec les conditions: $\gamma_0(x)=1$, $\gamma_1(x)=x$, $\gamma_r(x)\gamma_s(x)=(r, s)\gamma_{r+s}(x)$, $\gamma_r(x+y)=\sum_{s+t=r}\gamma_s(x)\gamma_t(y)$, $\gamma_s(\gamma_r(x))=\varepsilon_{s,r}\gamma_{sr}(x)$ pour $s>0$, $r>0$, $\gamma_r(xy)=r!\gamma_r(x)\gamma_r(y)$ pour x et y de degrés pairs $\geqq 2$, $\gamma_r(xy)=0$ pour x, y de degrés impairs et $r\geqq 2$; $(r, s)=(r+s)!/r!s!$, et $\varepsilon_{s,r}=\prod_{1\leq i\leq s-1}(ir, r-1)$. Les morphismes de la catégorie $\Gamma(\mathscr{R})$ sont définis de manière évidente. L'Auteur considère un entier premier p, donné une fois pour toutes, et se borne à la sous-catégorie $\Gamma_p(\mathscr{R})$ formée des A tels que, pour tout $n\geqq 1$, A_n soit un groupe cyclique infini ou d'ordre égal à une puissance de p.

Soit X un espace (ou un ensemble simplicial). On considère la cohomologie de X à coefficients dans un $A\in\Gamma_p(\mathscr{R})$. Pour $r\geqq 0$, $k\geqq 1$, $n\geqq 1$, on a une opération cohomologique $\mathfrak{p}_r$ qui envoie $H^{2k}(X; A_{2n})$ dans $H^{2rk}(X; A_{2rn})$, avec les propriétés suivantes: $\mathfrak{p}_0(u)=1$, $\mathfrak{p}_1(u)=u$, $\mathfrak{p}_r(u)\cup\mathfrak{p}_s(u)=(r, s)\mathfrak{p}_{r+s}(u)$, $\mathfrak{p}_r(u+v)=\sum_{s+t=r}\mathfrak{p}_s(u)\cup\mathfrak{p}_t(v)$, $\mathfrak{p}_s(\mathfrak{p}_r(u))=\varepsilon_{s,r}\mathfrak{p}_{sr}(u)$ pour $s>0$, $r>0$, $\mathfrak{p}_r(u\cup v)=r!\mathfrak{p}_r(u)\cup\mathfrak{p}_r(v)$ si u et v sont de bidegrés pairs, sinon $\mathfrak{p}_r(u\cup v)=0$ pour $r\geqq 2$; le signe $\cup$ désigne le cup-produit $H^k(X; A_n)\otimes H^{k'}(X; A_{n'})\rightarrow H^{k+k'}(X; A_{n+n'})$. De plus les $\mathfrak{p}_r$ jouissent des propriétés fonctorielles désirées vis-à-vis de X et de A.

En fait, il suffit de définir les $\mathfrak{p}_r$ lorsque $A=\Gamma(\Pi)$, Π étant un groupe cyclique $Z/\theta Z$, où l'entier θ est nul ou égal à une puissance de p; Γ désigne le foncteur classique d'Eilenberg-MacLane, et $\Gamma(\Pi)$ est bien dans la catégorie $\Gamma_p(\mathscr{R})$. Soit $\Gamma_n(\Pi)$ le groupe des éléments de degré $2n$, $\Gamma_1(\Pi)$ étant identifié à Π. Choisissons un générateur a de Π, ce qui fixe sur Π une structure d'anneau; alors $\gamma_n(a)$ est un générateur du groupe cyclique $\Gamma_n(\Pi)$, dont l'ordre est égal au produit $\theta\cdot[n, \theta^\infty]$, où $[n, \theta^\infty]$ désigne le p.g.c.d. de n et θ^q pour q grand. Le générateur $\gamma_n(a)$ définit sur $\Gamma_n(\Pi)$ une structure d'anneau, et on identifie $\Gamma_r(\Gamma_s(\Pi))$ à $\Gamma_{rs}(\Pi)$ en envoyant $\gamma_r(\gamma_s(a))$ en $\gamma_{rs}(a)$. Notons $\eta_r(\Pi)$ l'homomorphisme $\Gamma_r(\Pi)\rightarrow\Pi$ qui envoie $\gamma_r(a)$ en a. Les opérations $\mathfrak{p}_r$ jouissent des propriétés additionnelles suivantes:

(1) $\quad \eta_r(\Pi)\circ\mathfrak{p}_r(u)=u^r$ pour $u\in H^{2k}(X; \Pi)$,

où u^r désigne la puissance r-ième au sens du cup-produit défini dans $H^*(X; \Pi)$ par la structure d'anneau de Π; plus généralement:

(2) $\quad \eta_r(\Gamma_s(\Pi))\circ\mathfrak{p}_{rs}(u)=(\mathfrak{p}_s(u))^r=\mathfrak{p}_s(u^r)$

pour $u\in H^{2k}(X; \Pi)$.

De plus, si r est impair ou si θ est nul ou impair, on a

(3) $\quad \mathfrak{p}_r(u_1\cdots u_n)=\mathfrak{p}_r(u_1)\cdots\mathfrak{p}_r(u_n)$ pour $u_i\in H^{2k_i}(X; \Pi)$,

où les produits s'entendent dans $H^*(X; \Pi)$, resp. $H^*(X; \Gamma_r(\Pi))$. Par contre, si r et θ sont pairs, la différence des deux membres de (3) n'est pas nécessairement nulle; c'est un élément d'ordre 2 de $H^*(X; \Gamma_r(\Pi))$.

La définition des opérations $\mathfrak{p}_r$ met en oeuvre beaucoup de technique. On définit d'abord $\mathfrak{p}_r$ lorsque r est premier, et en choisissant un générateur a de $\Pi\approx Z/\theta Z$. Pour cela, on emploie la méthode de Steenrod [Comment. Math. Helv. **31** (1957), 195–218; MR **19**, 1069]: soit M un complexe de cochaînes, Z-libre, ayant une base formée d'un élément u de degré $2n$, et (si $\theta\neq 0$) d'un élément v de degré $2n+1$, tel que $\delta u=\theta v$; soit M^r la puissance tensorielle r-ième (r premier), dans laquelle le groupe G, cyclique d'ordre r, opère (à droite) naturellement, et soit W le G-complexe de chaînes classique, ayant une G-base formée d'éléments e_i (i entier $\geqq 0$) tels que

$$de_0=0,\ de_{2i+1}=(T-1)e_{2i},\ de_{2i+2}=\Big(\sum_{0\leq j\leq r-1}T^j\Big)e_{2i+1},$$

T désignant un générateur de G. Considérons le complexe $M^r\otimes_G W$, gradué comme suit: le degré de $x\otimes y$ est la différence des degrés de x et de y (c'est un complexe de cochaînes). La cochaîne

$$u^r\otimes e_0-\theta vu^{r-1}\otimes\Big(\sum_{0\leq j\leq r-1}jT^j\Big)e_1$$

est un $(2rn)$-cocycle modulo θ', avec $\theta'=\theta\cdot[r, \theta^\infty]$; conformément à la théorie de Steenrod, ce cocycle définit une opération cohomologique $H^{2n}(X; Z/\theta Z)\rightarrow H^{2rn}(X; Z/\theta' Z)$; si on identifie $Z/\theta' Z$ à $\Gamma_r(Z/\theta Z)$, on obtient l'opération $\mathfrak{p}_r$ cherchée. Lorsque l'entier premier r est distinct de p, on a $\theta'=\theta$, et $\mathfrak{p}_r$ est la puissance r-ième; on montre que $\mathfrak{p}_r\circ\mathfrak{p}_{r'}=\mathfrak{p}_{r'}\circ\mathfrak{p}_r$ pour r et r' premiers, ce qui permet de définir $\mathfrak{p}_r$ pour entier r quelconque. On vérifie que l'opération $\mathfrak{p}_r$ est indépendante du choix du générateur de $\Pi=Z/\theta Z$.

L'Auteur étudie aussi des opérations $\mathfrak{p}_r$ appliquées aux éléments de degré impair de $H^*(X; Z/\theta Z)$, mais elles s'expriment à l'aide des autres opérations connues. Le cas $r=2$ mérite une étude spéciale; il met en jeu le carré de Postnikov $H^q(X; \Pi)\rightarrow H^{2q+1}(X; \Gamma_2(\Pi))$.

Il ne fait aucun doute que l'Auteur aurait pu tirer de ses résultats une définition des opérations $\mathfrak{p}_r$ dans le cas plus général où l'anneau A des coefficients est un anneau gradué à puissances divisées, tel que A_n soit, pour chaque degré n, un groupe abélien de type fini, sans autre condition.
H. Cartan (Paris)

5475:

Yo, Ging-tzung. The construction for Steenrod's D_i in reduced powers of cohomology classes. Sci. Record (N.S.) 1 (1957), 223–225.

Let $\mathscr{P}^i$: $H^q(K; Z_p)\rightarrow H^{q+2i(p-1)}(K; Z_p)$ denote the Steenrod reduced power operations. The author gives an explicit formula for $\mathscr{P}^i$ in case K is a simplicial complex.
F. P. Peterson (Cambridge, Mass.)

5476:

Chang, Su-cheng. On intrinsic inequalities associated with certain continuous mappings and their application to fibre spaces. Sci. Record (N.S.) 2 (1958), 98–100.

The author and J. H. C. Whitehead [Quart. J. Math. Oxford Ser. (2) 2 (1951), 167–174; MR 13, 374] introduced a system of numerical invariants, called block invariants, associated with a polyhedron. In this note the author considers a map of polyhedra $f: X\rightarrow Y$ and obtains inequalities relating the block invariants of X and Y in the case when f induces cohomology epimorphisms and in the case when f induces cohomology monomorphisms. It is pointed out that the results lead to necessary conditions on the block invariants for a fibre map to admit a cross-section.
P. J. Hilton (Ithaca, N.Y.)

5477:

Dedecker, Paul. Cohomologie de dimension 2 à coefficients non abéliens. C. R. Acad. Sci. Paris 247 (1958), 1160–1163.

"On introduit une notion de système de coefficients

non abéliens (du type faisceau) qui permet de définir un ensemble de 2-cohomologie d'un espace (réduit au groupe usuel dans le cas abélien). Il s'insère dans une suite exacte et conduit à un obstacle dont la 'neutralité' est nécessaire et suffisante pour qu'on puisse procéder à 'l'extension' d'un espace fibré principal relativement à une extension de son groupe structural." (Author's summary) *Sze-tsen Hu* (Detroit, Mich.)

5478:

Barcus, W. D.; and Meyer, J.-P. The suspension of a loop space. Amer. J. Math. 80 (1958), 895–920.

Let Ω denote the space of loops on the topological space X based at x_0. Part I of the paper is devoted to a study of the homology suspension

$$\sigma:H_n(\Omega)\to H_{n+1}(X).$$

Starting with the canonical map k: $S\Omega\to X$ (where $S\Omega$ is the suspension of Ω) the authors replace k by a fiber map and show that the fiber of the resulting fibering is of the same homotopy type as the join $\Omega*\Omega$. The Serre homology sequence of this fibering is essentially the same as an exact sequence of G. W. Whitehead [Ann. of Math. (2) 62 (1955), 254–268; MR 15, 520], but extends Whitehead's sequence to one more dimension. Thus, several of Whitehead's results are improved by one dimension; in particular, it follows that a cohomology operation of type $(n, q; \pi, G)$, $q\leqq 3n$, is additive if and only if it is a suspension.

Part II of the paper applies the fibering of the first part to the calculation of the Postnikov invariants of $SK(\pi, n)=S\Omega K(\pi, n+1)$. It follows from the exact homotopy sequence of the fibering and from a result of G. W. Whitehead [Trans. Amer. Math. Soc. 83 (1956), 55–69; MR 18, 327] that $\pi_i(SK(\pi, n))=0$ if $i<n+1$ or $n+1<i<2n+1$, $\pi_{n+1}(SK(\pi, n))=\pi$ and $\pi_{2n+s+1}(SK)\approx H_{n+s}(\pi, n; \pi)$ for $0\leqq s\leqq n-1$. Hence, the first non-trivial Postnikov invariant for $SK(\pi, n)$ is $k^{2n+2}\in H^{2n+2}(K(\pi, n+1); \pi\otimes\pi)$, which is shown to equal $-b\cup b$, where $b\in H^{n+1}(K(\pi, n+1); \pi)$ is the basic class. Results are also obtained expressing the higher Postnikov invariants k^{2n+s+1} for $2\leqq s\leqq n$, but these are more complicated.

E. H. Spanier (Princeton, N.J.)

5479:

Chow, Sho-kwan. Homotopy groups and cup product of cohomology groups. Acta Math. Sinica 8 (1958), 200–209. (Chinese. English summary)

For any arcwise connected space X, let $\rho_m(X)$ denote the mth Betti number of X. If R is the field of rational numbers, $P_m(X)$ will denote the subspace of the vector space $H^m(X, R)$ generated by the elements of the form $a\cup b$ with $a\in H^i(X, R)$, $b\in H^j(X, R)$, $i+j=m$, and $0<i<m$. The dimension of $P_m(X)$ is denoted by $p_m(X)$. The following theorem is proved: If X satisfies the following conditions: (a) $\pi_1(X)=0$; (b) when $1<n\leqq N-1$, $H_n(X, R)=0$, $N\geqq 2$; then, for any integer l such that $2N-1\leqq l\leqq 3N-3$, the rank of $\pi_l(X)$ is equal to the sum

$$\sum_{\substack{m+m'=l+1\\ N\leqq m<m'\leqq 2N-2}}\rho_m(X)\rho_{m'}(X)+\rho_l(X)+\varepsilon_l(X)-p_{l+1}(X)-p_l(X),$$

where $\varepsilon_l(X)$ is equal to $(\rho_{\frac12(l+1)}(X)+1)\rho_{\frac12(l+1)}(X)/2$ when $l+1=0 \bmod 4$; to $(\rho_{\frac12(l+1)}(X)-1)\rho_{\frac12(l+1)}(X)/2$ when $l+1=2 \bmod 4$; and to 0 when $l+1=1 \bmod 2$.

Let u be an element of $H_m(X, R)$. If there exists a map $f:S^m\to X$ such that u is contained in $f_*(H_m(S^m, R))$, then u is said to be spherical. The following theorem is proved: If X satisfies the conditions in the theorem above, then: (a) Every element of $H(X, R)$ is spherical, $N\leqq l\leqq 2N-1$; (b) for any element u of $H(X, R)$, $2N\leqq l\leqq 3N-3$, u is spherical if and only if it is orthogonal to $P_m(X)$.

Sze-tsen Hu (Detroit, Mich.)

5480:

Chang, Su-cheng. On homotopy types and homotopy groups of polyhedra. I, II. Sci. Record (N.S.) 1 (1957), 205–213.

Following the work of J. H. C. Whitehead, Shiraiwa [Amer. J. Math. 76 (1954), 235–251; MR 15, 458] described a cohomology system for $A_n{}^3$-polyhedron, $n>3$, and showed it to be a complete invariant of homotopy type. In this paper the author continues his programme (initiated in the study of $A_n{}^2$-polyhedra) of obtaining computable invariants for $A_n{}^3$-polyhedra. In addition to the invariants he has already introduced [Bull. Acad. Polon. Sci. 4 (1956), 113–118; MR 19, 972] he defines two new invariants which he calls Φ_1-torsion and Φ_2-torsion. These, together with Betti numbers, torsions, block invariants, relative block invariants, characteristic polynomials and characteristic coefficients (see the paper referred to above) constitute a complete and independent set of homotopy type invariants for $A_n{}^3$-polyhedra of a particular sort, namely, those whose cohomology groups have 2-primary component of type $(k, k, \cdots, k)$ for some integer k.

The author also produces a formidable collection of 'simple' $A_n{}^3$-polyhedra, such that every homotopy type (of $A_n{}^3$-polyhedra of the restricted type) contains a unique 'normal form' consisting of a union of simple polyhedra. *P. J. Hilton* (Ithaca, N.Y.)

5481:

Heilbronn, H. On the representation of homotopic classes by regular functions. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 181–184.

Let A and B be any two open sets on the complex sphere. Consider all homotopy classes of continuous functions which map A into B. Is it true that each homotopy class contains a function meromorphic in A?

This question is answered, affirmatively or negatively, for various assumptions about the connectivity of A and B. *From the introduction*

5482:

Curtis, M. L. A note on Kosiński's r-spaces. Fund. Math. 46 (1958), 25–27.

A point x in a space X is called an "r-point" if x has arbitrarily small neighbourhoods U such that for each $y\in U$ there is a deformation retraction of $\bar U-y$ onto $\bar U-U$. X is an r-space if each of its points is an r-point, and if it is compact, metric and finite-dimensional. The author gives an example of a 4-dimensional finite polyhedron P^4 which is not an r-space, but which is such that its Cartesian product $P^4\times S^1$ with a 1-sphere is an r-space. He takes P^4 to be the suspension SM^3 of a 3-dimensional Poincaré manifold M^3, and shows that the second suspension S^2M^3 is an r-space. Every point of $P^4\times S^1$ has a neighbourhood like some point of S^2M^3, and so $P^4\times S^1$ is also an r-space. The two "vertices" of S^2M^3 each have arbitrarily small simplicial neighbourhoods with frontier homeomorphic to P^4, and so these frontiers are not r-spaces. Thus, two of the problems given by A. Kosiński [Fund. Math. 42 (1955), 111–124; MR 17, 654] are settled. It is still unknown whether or not the r-spaces S^2M^3, $P^4\times S^1$ are locally euclidean. *H. B. Griffiths* (Bristol)

5483:

Baiada, E. La sfera topologica ed il proprio interno. Matematiche, Catania **11** (1956), 107–110 (1957).

Let M be an n-sphere smoothly imbedded in E^{n+1}. Under the hypothesis that a certain function f on M has just two critical points, the author states that M bounds an $(n+1)$-cell. The author also states theorems concerning homeomorphisms and isotopies of regions on M of the form $a \leqq f \leqq b$. *P. A. Smith* (New York, N.Y.)

5484:

★Rattray, B. A. Generalizations of the Borsuk-Ulam theorem. Symposium internacional de topologia algebraica [International symposium on algebraic topology], p. 302. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

The paper contains statement and proof of the following theorems. (1) Any continuous map of the n-sphere into itself carries at least one pair of antipodal points into the same point, if its degree is even, or (2) at least one pair of antipodal points onto a pair of antipodal points if its degree is odd. — As a consequence, (3) any continuous map of the n-sphere into the projective n-space carries some pair of antipodal points into the same point. {It is obvious that this result holds only for $n>1$, although this is not stated in the paper.}

Statement and proof of (1) were first given by K. Borsuk [C. R. Soc. Sci. Varsovie **31** (1938), 7–12]; statement and proof of (2) were first given by the reviewer [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 394–399; MR **9**, 52], as the author added in proof; theorems 1, 2, 3, with essentially the same proof, were also given by J. W. Jaworowski and K. Moszyński [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 75–77; MR **18**, 226].
G. Hirsch (Brussels)

5485:

Heller, Alex. Twisted ranks and Euler characteristics. Illinois J. Math. **1** (1957), 562–564.

If Q is a finite group and G a finite right Q-module, Q is said to be an Artin-Tate group with respect to G provided that for each prime p which divides the order of G, the p-Sylow subgroup of Q is cyclic or a generalized quaternion group. Using results of Artin-Tate on the periodicity of the homology of such groups the author generalizes his previous results [Ann. of Math. (2) **60** (1954), 283–303; MR **16**, 276] on the fixed point set of a group of prime order operating on a complex to the case of an Artin-Tate group operating on a complex. The constructions are formulated in abstract abelian categories.
N. Stein (New Haven, Conn.)

5486:

Harary, Frank. On arbitrarily traceable graphs and directed graphs. Scripta Math. **23** (1957), 37–41 (1958).

The author considers the special Euler graphs called arbitrarily traceable by the reviewer. It is shown that such a graph has at most one cut point and that the maximal subgraphs without cut points are the prime subgraphs introduced by Baebler [Comment. Math. Helv. **27** (1953), 81–100; MR **15**, 50]. Finally, arbitrarily traceable directed graphs are discussed.
O. Ore (New Haven, Conn.)

5487:

Newman, Donald J. A problem in graph theory. Amer. Math. Monthly **65** (1958), 611.

The author gives a simple proof of the following special case of a theorem of Dirac's [G. A. Dirac, Proc. London Math. Soc. (3) **2** (1952), 69–81; MR **13**, 856; Thm. 3]: If G is a graph of order $2n$, and if each vertex of G is of degree $\geqq n$, then G contains a $2n$-circuit.
G. Sabidussi (New Orleans, La.)

5488:

Kotzig, Anton. Die Zerlegung eines endlichen regulären Graphen ungeraden Grades in zwei Faktoren. Časopis Pěst. Mat. **83** (1958), 27–32. (Slovak. Russian and German summaries)

Verf. beweist den Satz: Ein regulärer Graph G $(2n+1)$-ten Grades mit $2m$ Knotenpunkten lässt sich in Faktoren n-ten und $(n+1)$-ten Grades dann und nur dann zerlegen, falls in G ein System $\mathfrak{S}$ von m offenen Zügen mit folgenden Eigenschaften existiert: (1) jede Kante in G gehört genau zu einem Zuge von $\mathfrak{S}$; (2) jeder Zug von $\mathfrak{S}$ besitzt eine ungerade Anzahl von Kanten. *M. Fiedler* (Prague)

5489:

Čulík, Karel. Theorie der verallgemeinerten Konfigurationen. Práce Brn. Českoslov. Akad. Věd **29** (1957), 225–255. (Czech. German and Russian summaries)

Unter einer (verallgemeinerten) k-dimensionalen Konfiguration $\mathbf{M}$ wird der Inbegriff einer aus k $(\geqq 2)$ Elementen $M^{(i)}$ bestehenden geordneten Zerlegung $\bar{M}=\{M^{(1)}, \cdots, M^{(k)}\}$ einer nicht leeren Menge M und einer auf dem kartesischen Quadrat $M\times M$ definierten sogenannten Inzidenzfunktion f verstanden. Von dieser letzteren wird angenommen, dass sie nur die Werte 0, 1 annimmt und insbesondere für je zwei in demselben Element $M^{(i)}$ $(i=1, \cdots, k)$ liegende Punkte $x, y \in M^{(i)}$ stets gleich Null ist. Bezeichnung: $\mathbf{M}=\{f, M^{(i)}\}_{i=1}{}^k$. Die in dem Element $M^{(i)}$ liegenden Punkte werden Punkte von der Art i $(=1, \cdots, k)$ genannt. Seien $\mathbf{M}=\{f, M^{(i)}\}_{i=1}{}^k$, $\mathbf{N}=\{g, N^{(i)}\}_{i=1}{}^k$ k-dimensionale Konfigurationen. Eine Abbildung φ der Menge $M=\bigcup_{i=1}^k M^{(i)}$ auf die Menge $N=\bigcup_{i=1}^k N^{(i)}$, von der Beschaffenheit, dass $\varphi(M^{(i)})=N^{(i)}$ für $1\leqq i\leqq k$ und $f(x, y)=g(\varphi(x), \varphi(y))$ für alle $x, y \in M$ ist, wird K-homomorph genannt. Ist die Abbildung φ überdies schlicht, so wird sie als K-isomorph bezeichnet. Eine Verfeinerung $\bar{V}$ von $\bar{M}$ wird erzeugende Zerlegung der Konfiguration M genannt, wenn für je zwei Elemente $X, Y \in \bar{V}$ und alle Punkte $x \in X$, $y \in Y$ die Beziehung $f(x, y)=$konst erfüllt ist. Zu jeder erzeugenden Zerlegung $\bar{V}$ von $\mathbf{M}$ gehört eine auf $\mathbf{M}$ liegende k-dimensionale Faktorkonfiguration. Dieselbe ist durch die geordnete Zerlegung $\bar{\bar{M}}=\{\bar{M}^{(1)}, \cdots, \bar{M}^{(k)}\}$ von $\bar{V}$ und die Inzidenzfunktion $\bar{f}$ bestimmt, wobei jedes $\bar{M}^{(i)}$ aus den in $M^{(i)}$ enthaltenen Elementen von $\bar{V}$ besteht und der Wert der Funktion $\bar{f}$, für beliebige Elemente $X, Y \in \bar{V}$ durch die Formel: $\bar{f}(X, Y)=f(x, y)$, $x \in X$, $y \in Y$, definiert ist. Die in $\bar{M}^{(i)}$ enthaltenen Elemente von $\bar{V}$ stellen also die Elemente von der Art i $(=1, \cdots, k)$ in bezug auf die Faktorkonfiguration $\bar{\mathbf{M}}=\{\bar{f}, \bar{M}^{(i)}\}_{i=1}{}^k$ dar. Es wird gezeigt, dass die Konfiguration $\mathbf{M}$ ein K-homomorphes Urbild einer jeden auf ihr liegenden Faktorkombination darstellt und zugleich jedes K-homomorphe Bild der Konfiguration $\mathbf{M}$ mit einer auf ihr gelegenen Faktorkonfiguration K-isomorph ist. Auf Grund der obigen Begriffe wird eine inhaltsreiche Theorie aufgebaut, in der zahlreiche neue Begriffe eingeführt und ihre gegenseitigen Beziehungen besprochen werden. *O. Borůvka* (Brno)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 5299, 5483, 5567.

5490a:

Bompiani, Enrico. Su certi complessi quadratici e cubici di rette. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 213–219.

5490b:

Bompiani, Enrico. Complessi quadratici e cubici di spazi dotati di omologie armoniche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 371–377.

5491:

Jonas, Hans. Ein Transformationsprozess für Orthogonalnetze der Kugel mit Anwendungen auf spezielle Flächenklassen und ein Beitrag zur Theorie der zyklischen Strahlensysteme. Math. Nachr. **18** (1958), 64–95.

L'A., en étudiant les correspondances entre les réseaux orthogonaux d'une sphère pour lesquels les tangentes qui ne se correspondent pas se coupent (aux points y, z), envisage certaines surfaces qui admettent des déformations qui ne dépendent que des quadratures. Le cas spécial où y, z forment les réseaux conjugués correspondants est en connexion avec certaines systèmes de paraboloides [Jonas, S.-B. Berlin. Math. Ges. **34** (1935), 52–71]. Enfin l'A. trouve les réseaux conjugués dont les deux systèmes des tangentes forment des congruences cycliques. Il n'est pas possible de présenter une idée plus nette du travail sans la citation très étendue des calculs et des résultats partiels. *A. Švec* (Prague)

5492:

Godeaux, Lucien. Familles de quadriques attachées à des congruences W. Rev. Math. Pures Appl. **1** (1956), no. 3, 93–97.

5493:

Godeaux, Lucien. Une extension de la notion de congruences stratifiables. Math. Nachr. **18** (1958), 57–63.

Soient données deux suites de Laplace $\cdots, U_n, \cdots, U, V, \cdots, V_n, \cdots; \cdots, A, B, \cdots$ d'espace projectif S_r. Si $A \in [U_n U_{n-1} \cdots V_{n-1}]$ et $B \in [U_{n-1} U_{n-2} \cdots V_n]$, nous dirons que la congruence $[AB]$ est n-stratifiable à la congruence $[UV]$. Voici une construction géométrique: On considère dans S_{r+2n} une suite de Laplace $\bar{L}$ et deux espaces S_r, S_{2n-1} ne se rencontrant pas. La projection de $\bar{L}$ à partir de S_{2n-1} sur S_r est une suite de Laplace L et l'intersection de l'espace $[\bar{U}_n \cdots \bar{V}_n]$ avec S_r est une droite engendrant une congruence n-stratifiable à la congruence $[UV]$. La condition nécessaire et suffisante pour que les deux congruences précédentes soient n-stratifiables dans les deux sens est que $[\bar{U}_{2n} \cdots \bar{V}_{2n-1}]$ coupe S_{2n-1} en un point situé sur la droite $[U\bar{U}]$.
A. Švec (Prague)

5494:

Backes, F. Sur les surfaces pseudosphériques et leur extension en géométrie projective. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 457–465.

The author proves that if two congruences W of normals have a common focal nappe, then this surface is necessarily one of constant total curvature. Also, every pseudospherical surface is a focal nappe of a doubly infinite system of normal congruences W. A surface P is a common focal nappe of two congruences W of normals for which the intersection of the second focal planes constitutes a congruence conjugate to P. Each such surface P is a focal nappe of a doubly infinite system of congruences W such that the second focal points contained on a common line generate a congruence conjugate to the surface P. Finally, the author notes that, by means of the surfaces P, these theorems can be extended into projective differential geometry. *J. De Cicco* (Chicago, Ill.)

5495:

Papuc, D. Sur la théorie des hypersurfaces dans un espace axial à n dimensions. An. Şti. Univ. "Al. I. Cuza" Iaşi. Sect. I (N.S.) **3** (1957), 133–164. (Russian and Romanian summaries)

The author defines the n-dimensional axial space $I_{n-m}{}^n$ to be a Klein space in which the subgroup of projective transformations leaves an $(n-m)$-dimensional linear subspace I_{n-m} invariant. I_{n-m} is the axis of the space. The aim of the paper is the discussion of hypersurfaces in axial spaces. The chief tool of investigation is the method of projective normalization due to Norden. By imposing a sequence of conditions upon the considered hypersurface, the author determines a unique normalization of this hypersurface, characterized by the property of being axial and reciprocal. The chief result of the paper is the proof of an existence and unicity theorem, which states that the fundamental quantities a_{ij} and the connection object $G_{ij}{}^h$ of $I_{n-m}{}^n$ determine (up to an axial transformation) a hypersurface with an axial and reciprocal normalization, provided that the above quantities satisfy a set of restrictive requirements.
G. Soós (Debrecen)

5496:

Ruscior, Ştefania. La correspondance par plans tangents complètement parallèles entre deux surfaces réglées, dans S_4. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), 25–28. (Russian and Romanian summaries)

Two planes in a four-dimensional space are called completely parallel [semi-parallel], if their intersection is a line [a point] in the hyperplane at infinity. A correspondence between two ruled surfaces S and S^*, which maps the rectilinear generators of S into those of S^*, is said to be a correspondence by completely parallel tangent planes if the tangent planes at corresponding points are completely parallel. The paper contains a necessary and sufficient condition for the existence of such a correspondence. *G. Soós* (Debrecen)

5497:

Krasnodębski, R. The differential invariants of a curve in symplectic space. Prace Mat. **2** (1958), 299–308. (Polish. Russian and English summaries)

Let $[vw]$ be the symplectic scalar product of two vectors v, w in a symplectic linear space G_{2r}. Let $C(x(s))$ be a curve in G_{2r}. Consider the scalar products

$$K_{h|k} = \left[\frac{d^h x}{ds^h} \frac{d^k x}{ds^k}\right] \quad (h=1, \cdots, k-1).$$

If $K_{1|2}=1$, then s is the symplectic arc of C and $K_{h|k}$ are differential invariants of C. Taking the derivatives of $K_{1|2}$ one expresses easily $K_{h|k}$, $k \geq 3$ in terms of $K_{g-1|g}$, $g<k$ and their derivatives. The invariants $K_{g-1|g}$ ($g=3, 4 \cdots, 2r+1$) constitute the complete set of differential invariants of C. *V. Hlavatý* (Bloomington, Ind.)

5498:

Vranceanu, Gheorghe. Espaces de Riemann partiellement projectifs à métrique indéfinie. Math. Nachr. 18 (1958), 123–126.

Etant donné un espace de Riemann V_n à métrique indéfinie partiellement projective d'ordre $n-m-1$, la métrique peut être écrite sous la forme

$$ds^2=2dx^idx^{m+i}+2\frac{\partial f_p}{\partial x^i}dx^idx^{2m+p}+a_{\alpha\beta}dx^\alpha dx^\beta$$

($i\leqq m$; $\alpha, \beta>m$; $p\leqq n-2m$). Dans le cas d'ordre maximum ($n=2m$) on a la forme canonique

$$ds^2=2dx^idx^{m+i}+\varphi(x^idx^{m+i})^2+b_{\alpha\beta}dx^\alpha dx^\beta$$

où φ et $b_{\alpha\beta}$ dépendent seulement des variables x^α. Pour le cas où la métrique est définie positive voir le livre de l'auteur [Lectures on differential geometry, vol. II, Ed. Acad. R. P. Romåne, 1951; MR **16**, 1049; ch. I].

A. Švec (Prague)

5499:

Yasunaka, Kuniho. Four vertices theorems for surface curves and space curves. Yokohama Math. J. **5** (1957), 201–208.

Expressing the fundamental quadratic form $dS^2=g_{\mu\nu}(x^\lambda)dx^\mu dx^\nu$ of a Riemannian manifold as $dS^2=\omega_\mu{}^l(x^\lambda)\omega_\nu{}^l(x^\lambda)dx^\mu dx^\nu$, where $\lambda, \mu, \nu, l=1, \cdots, n$, the author notes that the differential equations for the geodesic curves of the second kind, namely

$$\omega_\lambda{}^l\left(\frac{d^2x^\lambda}{dS^2}+\Lambda_{\mu\nu}{}^\lambda\frac{dx^\mu}{dS}\frac{dx^\nu}{dS}\right)=0,$$

where $\Lambda_{\mu\nu}{}^\lambda=\Omega_l{}^\lambda(\partial\omega_\mu{}^l/\partial x^\nu)$, $\Omega_l{}^\lambda\omega_l{}^\mu=\delta_\mu{}^\lambda$, $\Omega_n{}^\lambda\omega_\lambda{}^m=\delta_n{}^m$, admit integrals of the form $\xi^l=a^lS+c^l$, with $a^la^l=1$; thus showing that the geodesic curves of the second kind behave, as for meet and join, like straight lines.

Since the ξ^l obey the same laws as Cartesian coordinates in Euclidean space, they are identified with these coordinates, and geometric entities are defined for them by analogy. For these entities, it is pointed out that analogues of geometric results in Euclidean space hold without alteration and that no new proofs are needed for these results.

In particular, the author defines curvature $1/R$ for the ξ^l, determines its relation to the ordinary curvature $1/r$, and for $1/R$ obtains an analogue of the four-vertices theorem that on each convex curve there are at least four points where the curvature has a local extreme value.

E. F. Beckenbach (Los Angeles, Calif.)

5500:

Sakakura, Eiichi. Isoperimetry on the surface. Yokohama Math. J. **5** (1957), 209–222.

See the first two paragraphs of the preceding review for notation and context. The author shows, for an arbitrary surface S, that in the ξ^l geometry the isoperimetric inequality $L^2-4\pi F\geqq 0$ holds between the length L of any closed rectifiable curve C on S and the area F enclosed by C on S. This is not to say, of course, that the inequality holds for the ordinary length and area; thus, a large spherical balloon can have an arbitrarily small circular aperture.

E. F. Beckenbach (Los Angeles, Calif.)

5501:

Greenspan, Donald. On vertices of space arcs. Ann. Mat. Pura Appl. (4) **44** (1957), 45–72.

Mukhopadhyaya [Bull. Calcutta Math. Soc. **1** (1909), 31–37; reprinted in Collected geometrical papers, Calcutta Univ. Press, 1929; pp. 13–20; see p. 15] stated that every plane oval of class C'' possesses at least four extrema of the curvature, where an oval may be defined as a simply closed curve with non-vanishing curvature. His result was extended by D. Fog [S-B. Preuss. Akad. Wiss. **5** (1933), 251–254] and W. C. Graustein [Trans. Amer. Math. Soc. **41** (1937), 9–23] to any simply closed curve with continuous curvature. Generalizations to ovals and simply closed curves which possess $2n$ vertices were found by W. Blaschke [Kreis und Kugel, de Gruyter, Berlin, 1930, p. 30], S. B. Jackson [Amer. J. Math. **62** (1940), 795–812; MR **2**, 158] and others. O. Haupt [Ann. Mat. Pura Appl. (4) **27** (1948), 293–320; MR **11**, 127], using a more general definition of vertex, avoiding previous differentiability assumptions, and using topological methods involving "Ordnungscharakteristik", showed that the four-vertices theorem and many of its generalizations were properly theorems of topology. Concerning three-dimensional arcs and curves, T. Takasu [Tôhoku Math. J. **39** (1934), 292–298; **41** (1935), 317–319], Graustein and Jackson [Bull. Amer. Math. Soc. **43** (1937), 737–741] exhibited the four-vertices theorem for certain restricted classes of space curves, and, by use of the concept of geodesic vertex, for certain curves on surfaces of constant curvature [Jackson, Amer. J. Math. **72** (1950), 161–186; MR **11**, 535]. By use of the concept of II-geodesic curvature of Takasu [Yokohama Math. J. **2** (1954), 81–94; MR **16**, 1053], K. Yasunaka [#5499 above] established four-vertices theorems for ovals with continuous II-geodesic curvatures on a surface and for a certain class of closed space curves with continuous II-geodesic curvatures. Takasu proved four-vertices theorems in the Lie higher circle geometry [Tôhoku Math. J. **28** (1933), 288–300], Möbius geometry and Laguerre geometry [Jap. J. Math. **10** (1933), 33–51; Proc. Phys.-Math. Soc. Japan (3) **16** (1934), 303–313].

In the present paper the author initiates a new theory of vertices of curves and arcs which will be of such generality as to allow valid results in n-dimensional Euclidean space, $n\geqq 3$. The concept of vertex used is a natural generalization of the one of Haupt cited above. Finally, the author sees that the methods used indicate a potential topological approach to the study of vertices which will generalize Haupt's two-dimensional techniques.

T. Takasu (Yokohoma)

5502:

Merza, J. L'introduction de la différentiation absolue dans l'espace affin. Publ. Math. Debrecen **5** (1958), 330–337.

It is shown how the covariant derivative of a vector on a surface in a flat unimodular affine three space can be obtained by projecting the derivative of the three dimensional vector on the tangent plane parallel to the affine normal. In the case of homogeneous affinities the projection is not parallel to the affine normal but to the position vector.

D. J. Struik (Cambridge, Mass.)

5503:

Hazanov, M. B. Deviation of a vector in parallel transport along a surface. Kabardin. Gos. Ped. Inst. Uč. Zap. **12** (1957), 15–16. (Russian)

If a vector is moved parallel to itself (in the sense of Levi Civita) along a closed curve C on a sufficiently small region of a surface, then after return to its original position it is turned over an angle equal to the total curvature of the area bounded by C [see e.g. A. Duschek and W. Mayer, Lehrbuch der Differentialgeometrie, I, Teubner, Leipzig-Berlin, 1930; p. 218].

D. J. Struik (Cambridge, Mass.)

5504:

Morrey, Charles B., Jr. The analytic embedding of abstract real-analytic manifolds. Ann. of Math. (2) **68** (1958), 159–201.

The author's contribution is to establish that a compact real-analytic ν-dimensional manifold without an analytic Riemannian metric can be imbedded analytically in Euclidean space. The problem is reduced to the case where the manifold is known to possess an analytic metric, and therefore an analytic imbedding because of results of Bochner, by showing for each point the existence of ν analytic functions on the manifold which have linearly independent gradients at that point. This is done by imbedding the real manifold in an open complex-analytic extension and then solving, on the basis of the author's familiar class P_2, a so-called $\bar{\partial}$-Neumann boundary value problem in the extension. This involves the usual apparatus of complex exterior differential forms, complex operators, and integral identities, as well as standard arguments from the theory of elliptic partial differential equations and their weak solutions. The essential features are a sufficiently refined treatment of behavior at the boundary and proof of the existence of non-trivial solutions of the boundary value problem. It could be mentioned that after the author had established his result, Grauert found another proof based on the theory of sheaves and avoiding boundary value problems [#5299 above].
P. R. Garabedian (Stanford, Calif.)

5505:

Rembs, Eduard. Die Weingartensche Funktion. Math. Nachr. **18** (1958), 96–98.

Soit $x(u, v)+tz(u, v)$ une déformation infinitésimale de la surface π: $x=x(u, v)$; alors $dz=y\times dx$ et $\varphi=(yn)$ ($n(u, v)$ est le vecteur unitaire normal) s'appelle la fonction de Weingarten. L'A. donne une autre démonstration du théorème [v. Math. Nachr. **16** (1957), 130–134; MR **19**, 675]: Soit π une surface convexe possédant une projection 1-1 sur un domaine G d'un plan, alors $\varphi=0$ à la frontière de G implique $\varphi=0$ dans G et π est indéformable.
A. Švec (Prague)

5506:

Šor, L. A. Bending of convex polyhedra with boundary. Mat. Sb. N. S. **45(87)** (1958), 471–488. (Russian)

Let P be a convex polyhedral surface in E^3 homeomorphic to E^2 whose boundary L is a closed Jordan polygon. Using the methods of A. D. Alexandrov, the paper determines all deformable P. The exact conditions are too involved to be stated here, the following remarks will provide a rough idea.

Let $\bar{P}$ be the boundary of the convex closure of P. Then $\bar{P}-P$ is homeomorphic to a closed disk and has no vertices in the interior. It can therefore be developed on the plane yielding a polygonal region Q covering (in general) parts of the plane more than once. Denote the boundary of Q by L_Q, and let $\tilde{Q}$ be the pointset covered by Q. Denote the points of L_Q corresponding to vertices of $\bar{P}$ lying on L as vertices of type A. The deformable P fall into three classes: 1) those which do not have vertices of type A; 2) those which have exactly one vertex of type A, where the corresponding point of $\tilde{Q}$ is not a multiple point of L_Q and is an actual vertex of the convex closure of $\tilde{Q}$, plus an extra condition on the corresponding vertex of $\bar{P}$; 3) polygons with several vertices of type A, such that the corresponding points of $\tilde{Q}$ are not multiple points of L_Q, and some extra conditions on the supporting lines at one of these vertices as points of Q.
H. Busemann (Cambridge, Mass.)

5507:

Pogorelov, A. V. On a transformation of isometric surfaces. Dokl. Akad. Nauk SSSR **122** (1958), 20–21. (Russian)

The paper contains a most surprising remark with strong implications.

Let $x_0, \cdots, x_3$ be cartesian coordinates in E^4. Denote by e_0 the unit vector in the direction of the x_0-axis. Interpret $x_0, \cdots, x_3$ also as the Weierstrass coordinates of a point x in an elliptic space EL^3 with curvature 1 or in a hyperbolic space H^3 with curvature -1.

Assume in EL^3 there are two intrinsically isometric surfaces $x=x^{(i)}(u, v)$, $i=1, 2$, where points with the same parameter values correspond under the isometry. In E^4 define the two surfaces

$$y=\frac{x^{(i)}(u, v)-e_0(x^{(i)}(u, v)\cdot e_0)}{e_0\cdot(x^{(1)}(u, v)\pm x^{(2)}(u, v))}\quad (i=1, 2).$$

They lie in the E^3: $x_0=0$ and are intrinsically isometric as surfaces in E^3. The analogous statement for H holds when the minus in the numerator is replaced by plus and the scalar product $a\cdot b=-a_0b_0+\sum_{i=1}^{3} a_i\cdot b_i$. Conversely, with two intrinsically isometric surfaces in E^3 there are associated two intrinsically isometric surfaces in EL^3 or H^3. Moreover, if a given pair is not only intrinsically isometric but congruent, then the corresponding pair is congruent. If the surfaces in EL^3 or H^3 are convex, then so are the corresponding surfaces in E^3. Consequently the fact that two intrinsically isometric closed convex surfaces in E^3 are congruent implies without further proof the corresponding result for EL^3 and H^3.
H. Busemann (Cambridge, Mass.)

5508:

Pogorelov, A. V. On the regularity of convex surfaces with a regular metric in spaces of constant curvature. Dokl. Akad. Nauk SSSR **122** (1958), 186–187. (Russian)

The author generalizes his well-known result for convex surfaces in E^3 to convex surfaces in three-dimensional spaces of constant curvature:

If the coefficients $g_{ik}(u^1, u^2)$ of a (not necessarily complete) convex surface in a space of constant curvature are of class c^k, $k\geqq 4$, and if the Gauss curvature is positive and greater than the curvature of the space, then the surface itself is at least of class $k-1$ (analytic if the g_{ik} are analytic).

The main steps of the long, only briefly sketched, proof are exactly the same as in E^3. In particular, the monotypy of caps among caps, i.e., the congruence of two intrinsically isometric convex caps, is used.
H. Busemann (Cambridge, Mass.)

5509:

Hsiung, Chuan-Chih. A uniqueness theorem on two-dimensional Riemannian manifolds with boundary. Michigan Math. J. **5** (1958), 25–30.

Theorem: Let M_2 and M_2^* be two oriented two-dimensional Riemannian manifolds of class C^3 imbedded in a Euclidean space E_{N+2} of dimension $N+2$ ($N>0$) with boundaries C and C^*, respectively, and with positive Gaussian curvatures in every normal direction. Suppose that there exists an orientation-preserving differentiable homeomorphism H of the manifold M_2 onto the manifold M_2^* such that at corresponding points the manifolds M_2 and M_2^* have parallel tangent planes and equal sums of the principal radii of curvature associated with every

common normal direction. If the homeomorphism H restricted to the boundary C is a translation carrying the boundary C onto the boundary C^*, then the homeomorphism H is a translation carrying the whole manifold M_2 onto the whole manifold M_2^*. For $N=1$ the theorem was previously proved by the author [Math. Z. **64** (1956), 41–46; MR **17**, 657]. *L. A. Santaló* (Buenos Aires)

5510:

Kawaguchi, Akitsugu; and Laugwitz, Detlef. Remarks on the theory of Minkowski spaces. Tensor (N.S.) **7** (1957), 190–199.

This consists of an exchange of letters between the two authors relating to their respective points of view in dealing with Minkowski and Finsler spaces. The papers whose contents are compared are (i) Kawaguchi, Tensor (N.S.) **6** (1956), 165–199 [MR **18**, 931] and (ii) D. Laugwitz and E. R. Lorch, Amer. J. Math. **78** (1956), 889–894 [MR **18**, 495]. *E. T. Davies* (Southampton)

PROBABILITY

See also 5591, 5696, 5697, 5701, 5720.

5511:

Nakamura, Masahiro. On the theory of independent random variables. Mem. Osaka Univ. Lib. Arts Ed. Ser. B. **4** (1955), 46–50.

The algebra of bounded random variables on a probability space is axiomatized as a commutative W^* algebra with a distinguished normal state. Independence of random variables is translated into measures being expressible as direct products of other measures. The point of view and the results are not new.

J. Feldman (Berkeley, Calif.)

5512:

Schweizer, Berthold; et Sklar, Abe. Espaces métriques aléatoires. C. R. Acad. Sci. Paris **247** (1958), 2092–2094.

Comparison of the spaces with stochastic metric of K. Menger [Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 535–537; MR **4**, 163] and A. Wald [ibid. **29** (1943), 196–197; MR **4**, 220]. *H. P. McKean, Jr.* (Cambridge, Mass.)

5513:

Massé, Pierre. Sur une généralisation de la méthode du pari d'Émile Borel. C. R. Acad. Sci. Paris **247** (1958), 1829–1830.

The subjective probability associated with an event X if $P(X)$. Let A be another event with probability a. The method of bets consists in letting the individual bet for A if $P(X)<a$, for X if $P(X)>a$ [E. Borel: Valeur pratique et philosophie des probabilités, Gauthier-Villars, Paris, 1939; ch. V, § 48, pp. 84–86]. Let I be the information which is used to estimate $P(X)$. The method of Borel holds if I is very great. If I is zero (no information) the minimax principle of Wald and Savage should be used. Bet for A with probability a and for X with probability $1-a$ (mixed strategy). Let $P_0(X)$ be the a priori probability of X. $P(X)=f(I, P_0(X))$. Let $b(I)$ be the value of the a priori probability for which the a posteriori probability has the value a: $a=f(I, b)$. Also let the event Y have $P_0(X)$, and the event $B, b(I)$. The procedure is: To bet for Y (hence for X) with probability $1-b$ and for B (hence for A) with probability b. If X is more probable than A, $b(I)\to 0$ with I and in the contrary case $b(I)\to 1$. Hence for large I Borel's procedure is valid.

G. Tintner (Lisbon)

5514:

Scapaticci, Augusto. Rette mediali e rette di minima distanza direzionale. Rend. Mat. e Appl. (5) **17** (1958), 35–81.

Let $\omega(x)$ be a probability density. The author studies carefully the solutions ρ of the equation $\int_\rho^\infty x\omega(x)dx=E(x)/2$. $E(x|x-y|)$ is an extremum for $y=\rho$. He investigates also the extrema of $E(x|x-y|^2)$. He is also concerned with the median regression curve, earlier investigated by J. Bejar [Trabajos Estadist. **7** (1956), 141–158; MR **18**, 771] and related problems.

L. Schmetterer (Berkeley, Calif.)

5515:

Révész, Pál. On the convergence of sequences of random variables (a remark on a problem of A. Prékopa). Magyar Tud. Akad. Mat. Kutató Int. Közl. **2** (1957), 51–58. (Hungarian and Russian summaries)

Let $\{\xi_n\}$ and $\{\eta_n\}$ be two sequences of random variables and let ζ_n denote the sum $\xi_n+\eta_n$. We assume that ξ_n and η_n are independent for all values of n. The author studies the relations among various hypotheses, in extension of the work of A. Prékopa [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 191–198; Publ. Math. Debrecen **4** (1956), 410–417; MR **20** #332 **19**, 891].

From the author's summary

5516:

Zaremba, S. K. Note on the central limit theorem. Math. Z. **69** (1958), 295–298.

The following result is proved: X_k $(k=1, 2, \cdots)$ are independent random variables with respective distribution functions $F_k(x)$. Let the following conditions be satisfied: i) For some non-negative integer p and for a suitably chosen sequence $\{a_n\}$ $(a_n>0)$, and for every positive η,

$$\lim_{n\to\infty} a_n^{-p}\sum_{k=1}^n \int_{|x|>\eta a_n} |x|^p dF_k(x)=0;$$

and (ii)

$$\lim_{n\to\infty} a_n^{-2}\sum_{k=1}^n \left\{\int_{|x|\le\eta a_n} x^2 dF_k(x)-\left(\int_{|x|\le\eta a_n} x dF_k(x)\right)^2\right\}=1.$$

If

$$b_{n,k}=\int_{|x|\le a_n} x dF_k(x),$$

the distribution of the random variable

$$S_n=a_n^{-1}\textstyle\sum_{k=1}^n (X_k-b_{n,k})$$

tends to the normal distribution with mean zero and unit variance. Further, if $p>0$, the moments of S_n up to the order p converge to the corresponding moments of the limiting distribution. By an extension of the argument used in proving the above result, the author also derives conditions for the central limit theorem for m-dependent random variables, which include earlier results as special cases. {The author's opening remark that "The central limit theorem for independent variables, as proved in its whole generality by Feller, is presently shown to be a fairly simple consequence of a weak form of the second limit theorem" is misleading. In this paper, only the sufficiency of the conditions is discussed, and the statement quoted does not apply to the necessity part of Feller's results.} *G. Kallianpur* (East Lansing, Mich.)

5517:

Fortet, Robert; et Mourier, Edith. Lois des grands nombres pour des fonctions aléatoires à valeurs dans un espace de Banach. C. R. Acad. Sci. Paris 247 (1958), 1288–1289.

This paper is a summary of results of a longer paper to appear.

Let m be a probability measure on a σ-algebra $\mathscr{S}$ of subsets of $\mathscr{U}$. Let $\mathscr{T}$ be the real line, equipped with Lebesgue measure L. Let x be a function from $\mathscr{U}\times\mathscr{T}$ to a separable Banach space $\mathscr{X}$. x is called $\mathscr{D}$-measurable (*) if, for each $x^*\in\mathscr{X}^*$, the function $(u, t)\to\langle x^*, x(u, t)\rangle$ is measurable with respect to $m\times L$. x is called strongly continuous in probability (**) if, for $t\in\mathscr{T}$, ε, $\mu>0$, there exists $\delta>0$ such that

$$m\{u|\ \|x(u, t+h)-x(u, t)\|>\varepsilon\}<\mu,$$

for all $|h|<\delta$ (presumably some assumption was intended to guarantee measurability of $x(\cdot, t)$). Theorem: (**) implies (*), and, conversely, (*), together with strict stationarity of x, implies (**). Theorem (a. e. law of large numbers): Let x be $\mathscr{D}$-measurable, strictly stationary, and $\int\|x(u, t)\|dm(u)<\infty$ for all t. Then there is a function Y from $\mathscr{U}$ to $\mathscr{X}$ such that $\langle x^*, Y(\cdot)\rangle$ is measurable for each $x^*\in\mathscr{X}^*$, and such that

$$\lim_{t\to+\infty} t^{-1}\int_0^t x(u, \tau)d\tau=Y(u)$$

for a.e. u. Theorem (mean law of large numbers): Let x be $\mathscr{D}$-measurable, strictly stationary, and $\int\|x(u, t)\|^\alpha\cdot dm(u)<\infty$, α being a fixed real number $\geqq 1$. Then there exists a function Y from $\mathscr{U}$ to $\mathscr{X}$, measurable as before, with $\int x(u, t)dm(u)=\int Y(u)dm(u)$ for all t, and

$$\lim_{t\to+\infty}\int\left\|t^{-1}\int_0^t x(u, \tau)d\tau-Y(u)\right\|^\alpha dm(u)=0.$$

J. Feldman (Berkeley, Calif.)

5518:

Fisz, M. A limit theorem for empirical distribution functions. Studia Math. 17 (1958), 71–77.

Proof of the theorem announced in Bull. Acad. Polon. Sci. Cl. III 5 (1957), 695–698 [MR 19, 693].

J. Wolfowitz (Ithaca, N.Y.)

5519:

Kiefer, J.; and Wolfowitz, J. On the deviations of the empiric distribution function of vector chance variables. Trans. Amer. Math. Soc. 87 (1958), 173–186.

Let F be a distribution function (d.f.) in E^m and let S_n be the empiric d.f. of n independent random variables having d.f. F. Denote by x a generic point in E^m and put $D_n=\sup_x|S_n(x)-F(x)|$. The authors prove that for each m there exist positive constants c_0 and c such that (*) $G_n(r)=P\{D_n<r/n^{\frac{1}{2}}\}>1-c_0e^{-cr^2}$ holds for all n, all F and all $r>0$. They also prove the existence of a d.f. G (depending on F) such that $G_n(r)\to G(r)$ as $n\to\infty$ at every continuity point of G, and similar results. For $m=1$, (*) was proved [Dvoretzky, Kiefer and Wolfowitz, Ann. Math. Statist. 27 (1956), 642–669; MR 18, 772] with $c=2$. The method of the quoted paper cannot be applied for $m>1$ and a more direct approach is developed. The authors remark that there are F in E^2 for which (*) does not hold with $c=2$. They give numerical lower bounds for c for $m\geqq 2$.

A. Dvoretzky (Jerusalem)

5520:

Nasr, S. K. A law of large numbers for abstract random variables. Metrika 1 (1958), 89–98.

Let $\mathfrak{X}$ be a set and let $D=\{d_t, t\in T\}$ be a separating family of pseudo-metrics on $\mathfrak{X}$. Let $X_1, X_2, \cdots, X_n$ be random variables taking their values in $\mathfrak{X}$. A random variable Z_n such that $nd_t(Z_n, a)\leqq\sum_i d_t(X_i, a)$ for every $a\in\mathfrak{X}$ and $t\in T$ is called an arithmetic mean of $\{X_i\}$ $(i=1, 2, \cdots, n)$. [See S. Doss, Bull. Sci. Math. (2) 73 (1949), 48–72; MR 11, 190; and S. K. Nasr, Publ. Inst. Statist. Univ. Paris 5 (1956), 33–42; MR 19, 325.] It is shown that for certain families D and certain sequences of independent random variables there are elements a_n of $\mathfrak{X}$ for which $d_t(Z_n, a_n)$ converges to zero on a set, depending on t, of probability unity. Under supplementary conditions a_n may be taken equal to the Shafik Doss expectation of Z_n. The prototype of families D for which the results are proven is the family of pseudo-norms of the weak topology of a Banach space.

L. M. LeCam (Berkeley, Calif.)

5521:

Rogozin, B. A. Some problems in the field of limit theorems. Teor. Veroyatnost. i Primenen. 3 (1958), 186–196. (Russian. English summary)

This paper consists of two independent parts. In the first the author obtains the following result (theorem 1): Let $\{\xi_i\}$ be a sequence of independent random variables with cumulative distribution functions (c.d.f.'s) $F_i(x)$ such that $F_i(x)=0$ if $x\leqq 0$ and $F_i(x)=\min[0, 1-\{c_i+\alpha_i(x)\}x^{-\alpha}]$ if $x>0$, where $|\alpha_i(x)|\leqq\alpha(x)\to 0$ as $x\to\infty$ and $0<c'<c_i<c''<\infty$ for $i=1, 2, \cdots$, and $0<\alpha<1$. Then

$$\text{prob}[\xi_1+\cdots+\xi_n<xB_n]\to F_\alpha(x)\ (n\to\infty),$$

where $B_n=(\sum_1^n c_i)^{1/\alpha}$ and $F_\alpha(x)$ has the characteristic function (ch.f.)

$$f_\alpha(t)=\exp[-\Gamma(1-\alpha)\cos\tfrac{1}{2}\pi\alpha\cdot|t|^\alpha\{1-i\tan\tfrac{1}{2}\pi\alpha\cdot\text{sign}\ t\}].$$

[The author states the formula for $F_i(x)$ incorrectly, since he fails to exclude negative values, but his proof is not affected.] He gives also a similar theorem for symmetrical distributions.

In his second part he obtains an explicit bound corresponding to a convergence theorem due to F. J. Dyson [Canadian J. Math. 5 (1953), 554–558; MR 15, 215], as follows (theorem 4): Let $\mathscr{F}_1$ be a family of absolutely continuous c.d.f.'s $F_1(x)$ that have uniformly bounded densities $p_1(x)<c<\infty$ $(-\infty<x<\infty)$ and are such that

$$|F_1(-x)+1-F_1(x)|<C(x)\ (x>0),$$

where $C(x)\to 0$ as $x\to\infty$. (c and $C(x)$ are to be the same for every F_1 in $\mathscr{F}_1$.) If $|f_1(t)-f_2(t)|<\varepsilon$ when $|t|<T$, where $f_1(t)$ is the ch.f. of an F_1 in $\mathscr{F}_1$ and $f_2(t)$ is that of some c.d.f. F_2 (possibly not in $\mathscr{F}_1$), then

$$|F_1(x)-F_2(x)|<N(\beta)[\varepsilon+16c(\pi\beta T)^{-1}]+3\beta\ (-\infty<x<\infty)$$

for any β with $0<\beta<1$, where $N(\beta)$ is a function depending on $\mathscr{F}_1$.

H. P. Mulholland (Exeter)

5522:

Minlos, R. A. Continuation of a generalized random process to a completely additive measure. Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 439–442. (Russian)

The author treats stochastic processes based on linear spaces, following I. M. Gel'fand [Dokl. Akad. Nauk SSSR (N.S.) 100 (1955), 853–856; MR 16, 938]. At the base is a real linear topological space E with conjugate space E'. For each n-tuple $\phi_1, \cdots, \phi_n$ of elements of E, and element F of E', the n-tuple $F(\phi_1), \cdots, F(\phi_n)$ is assigned a probability distribution. Corresponding to the linearity of E, a condition equivalent to the condition that

$$\text{Prob}\{F(\phi_1+\phi_2)=F(\phi_1)+F(\phi_2)\}=1$$

is imposed, as well as the condition of continuity that to every $\varepsilon>0$ corresponds a neighborhood U of the zero element of E and a positive constant A such that, if $\phi\in U$, $\mathrm{Prob}\{|F(\phi)|>A\}<\varepsilon$. Suppose further that (1) the topology of E is determined by a sequence of inner products $\{(\phi_1, \phi_2)_n, n\geqq 1\}$ for which $(\phi, \phi)_n$ increases with n, and (2) the sum of the squares, using the nth norm, of the principal axis lengths of the ellipsoid $(\phi, \phi)_{n+1}=1$ converges. It is shown that then the finite-dimensional assigned distributions can be extended to determine a completely additive measure on E', so that $\{F(\phi), \phi\in E\}$ is a stochastic process in the usual sense, with parameter space E. The proof is based on an apparently hitherto unpublished result of Erohin (unproved here) giving a necessary and sufficient condition that the above measure extension is possible under the hypothesis that the above condition (1) but not necessarily (2) is satisfied. The author states, however, that both conditions are necessary if all given distributions can be extended. The author's result is closely related to one of Winkelbauer [Czechoslovak Math. J. **6(81)** (1956), 517–521; MR **19**, 778], who supposes that E is the space of infinitely differentiable functions on k-space, with compact support.

J. L. Doob (Urbana, Ill.)

5523:

Freund, John E. Some results on recurrent events. Amer. Math. Monthly **64** (1957), 718–720.

This paper develops and illustrates a fundamental theorem for recurrent events for the non-discrete case. An event E is said to be recurrent if there is a rule which stipulates whether or not E occurs at a time t, and starting from an occurrence of E the future is identical to (probabilistically speaking), and independent of, the past. Let the element of probability that E occurs at t be $u(t)dt$ and let $f(t)dt$ be the element of probability that E occurs at t for the first time. Defining

$$U(\theta)=\int_0^\infty u(y)e^{\theta y}dy \text{ and } F(\theta)=\int_0^\infty F(x)e^{\theta x}dx,$$

the author shows that $F(\theta)=U(\theta)/(1+U(\theta))$. He uses this result to give still another elegant proof that waiting times in the Poisson process are exponentially distributed. The paper terminates with a derivation of the waiting time distribution for two independent Poisson processes to have the same number of occurrences.

H. Raiffa (Cambridge, Mass.)

5524:

Hennequin, Paul-Louis. Processus en cascade à n dimensions et problèmes de moments. C. R. Acad. Sci. Paris **247** (1958), 857–859.

Given a nice birth and death process on the non-negative integers with generator (infinitesimal matrix) A, Karlin and McGregor [Trans. Amer. Math. Soc. **85** (1957), 489–546; MR **19**, 989] gave the eigen-differential expansion for the transition function

$$(1)\quad p_{ij}(t)=\pi_j\int_0^{+\infty} e^{-xt}Q_i(x)Q_j(x)\psi(dx) \quad (i,j=0,1,2,\cdots;t\geqq 0),$$

in which π_j is a positive weight, Q_i is a polynomial of degree i, $Q.\,(=Q_i$ as a function of $i)$ solves $-xQ.=AQ.$, ψ is a non-negative Borel measure, and

$$(2)\qquad \int_0^{+\infty} Q_iQ_jd\psi=\begin{cases}\pi_i^{-1} & (i=j)\\ 0 & (i<j).\end{cases}$$

The author sketches the generalizations of (2) for birth and death processes on the lattice points $i=(i_1, \cdots, i_n)$ $(0\leqq i_1, \cdots, i_n)$ in $n(\geqq 2)$ dimensional space. The point of the π_j's figuring in (2) for $n=1$ is that A is a symmetric operator in the Hilbert space $l^2(\pi)$. When $n\geqq 2$, no such Hilbert space exists in general, and that forces the introduction of the two (dual) kinds of polynomials

$$(3)\qquad Q_i,\ Q_i^*(i\geqq 0),\ AQ.=xQ.,\ A^*Q.^*=xQ.^*$$

in terms of which the eigen-differential expansion assumes the form

$$(4)\qquad p_{ij}(t)=\int e^{xt}\mathbf{Q}_i(x)\psi(dx)\mathbf{Q}_j^*(x) \quad (i, j\geqq 0; t\geqq 0),$$

where $\mathbf{Q}$ and $\mathbf{Q}^*$ are vectors with entries Q and Q^* of a certain dimension $m\leqq+\infty$, ψ is a Borel measure from R^1 to $m\times m$ matrices and $\mathbf{Q}\psi\mathbf{Q}$ is the inner product of the vectors $\mathbf{Q}$ and $\psi\mathbf{Q}$. *H. P. McKean, Jr.* (Cambridge, Mass.)

5525:

Blackwell, David; and Koopmans, Lambert. On the identifiability problem for functions of finite Markov chains. Ann. Math. Statist. **28** (1957), 1011–1015.

Let $x_1, x_2, \cdots$ be a stationary Markov process with N states and an irreducible and aperiodic transition matrix M. For an arbitrary real function ϕ defined on the set of all states of the process, let $y_n=\phi(x_n)$ $(n=1, 2, \cdots)$.

The authors show the following. There exists a positive integer $J\leqq 2N^2+1$ depending only on N and ϕ such that the joint distribution of $y_1, y_2, \cdots$ is determined by the joint distribution of $y_1, \cdots, y_J$. Moreover, under mild restrictions on M, they find sets of invariants which determine the former distribution, and have a certain completeness property, in the following cases: (a) ϕ has only two values, one of which is assumed at only a single state; (b) $N=4$ and ϕ has only two values each of which is assumed on two states. The respective invariants are: (a) coefficients of certain polynomials determined by M, (b) probabilities that subsets of $(y_1, \cdots, y_4)$ assume 8 combinations of values specified in the paper.

H. M. Schaerf (St. Louis, Mo.)

5526:

Sarymsakov, T. A. On inhomogeneous Markoff chains. Dokl. Akad. Nauk SSSR **120** (1958), 465–467. (Russian)

Following Kolmogorov's definition of the "ergodic principle" [Uspehi Mat. Nauk **5** (1938), 52–56] the author defines the corresponding "uniform" concept as requiring uniform convergence with respect to the two initial points and the terminal set. A necessary and sufficient condition is announced for this uniform ergodic principle. Roughly stated it requires the Kolmogorov (sufficient) condition to apply to all increasing subsequences of time parameters.

K. L. Chung (Syracuse, N.Y.)

5527:

Flatto, L. A problem on random walk. Quart. J. Math. Oxford Ser. (2) **9** (1958), 299–300.

A particle moves on a d-dimensional lattice, taking unit steps. The probability that the particle moves $+1$ or -1 along the jth axis is $(1-\varepsilon/i)/2d$ or $(1+\varepsilon/i)/2d$, where i is the jth coordinate before the move, $|\varepsilon|<1$, $j=1, \cdots, d$. If $i=0$, the probabilities are $1/2d$. It was shown by Gillis [same J. **7** (1956), 144–152; MR **20** #2794] that the random walk is [is not] recurrent if $\varepsilon>[<]$ $(d-2)/2d$. The present paper shows that the walk is recurrent if $\varepsilon=(d-2)/2d$. The argument depends on the estimate $\gamma_N^{(1)}\geqq KN^{-\frac{1}{2}+\varepsilon}$, where $\gamma_N^{(1)}$ is the probability in the one-dimensional case that the particle is at its starting point after $2N$ steps.

T. E. Harris (Santa Monica, Calif.)

5528:

Foster, Caxton; and Rapoport, Anatol. The case of the forgetful burglar. Amer. Math. Monthly 65 (1958), 71–76.

This paper investigates a random walk problem in which each previously occupied point becomes an absorbing site. Starting from an arbitrary site of an infinite one-dimensional grid of sites, the walker takes with equal probability one or two steps in either direction. When a site previously occupied is re-entered, the walk terminates. The authors derive the frequency distribution of walk lengths.

M. Dresher (Pacific Palisades, Calif.)

5529:

Takács, Lajos. On random walk problems. Magyar Tud. Akad. Mat. Kutató Int. Közl. 2 (1957), 81–90. (Hungarian. Russian and English summaries)

New simple derivations are given for the known solutions of the random walk problem. This problem considers the motion of a particle on a straight line, which starts at $x=0$ and moves a unit distance in each step either to the right or to the left with the same probability of $\frac{1}{2}$, coming to η_n after n steps. The random walk with absorbing barriers at $\pm a$ is subject to the further condition that if the particle reaches the points $+a$ or $-a$ the motion is stopped. For the ordinary random walk, the following limiting distribution is known:

$$\lim_{n\to\infty} P\{\max(|\eta_1|, |\eta_2|, \cdots, |\eta_n|)<n^{\frac{1}{2}}z\}=F(z),$$

where $F(z)=0$ if $z\leqq 0$ and

$$(1)\quad F(z)=\sum_{k=-\infty}^{+\infty}(-1)^k[\Phi[(2k+1)z]-\Phi[(2k-1)z]]$$

if $z>0$, with $\Phi(z)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{z}\exp(-\frac{1}{2}y^2)dy$. For the random walk with absorbing barriers

$$(2)\quad P\{\eta_n^*=x\}=\sum_{j=-\infty}^{+\infty}[P\{\eta_n=4ja+x\} - P\{\eta_n=(4j+2)a-x\}]$$

if $x\neq\pm a$. The author gives first a simple and straightforward proof for (2), then he shows that from (2) the validity of (1) follows. Well-known alternative forms of (1) and (2) are also derived in simple ways.

Zoltán Bay (Washington, D.C.)

5530:

Cohen, J. W. The full availability group of trunks with an arbitrary distribution of the inter-arrival times and a negative exponential holding time distribution. Simon Stevin 31 (1957), 169–181.

The telephone traffic situation with the following properties is considered: there are n trunks; the inter-arrival times are independent with some arbitrary distribution; the holding times are independent with an exponential distribution, a new call may use any idle trunk; and if the trunks are all busy, any new call is either discarded or fed to some overflow trunks. Using the theory of Markov chains plus some interesting combinatorial manipulations, the author gives explicit formula for the equilibrium probability that a new arrival will meet j busy trunks and the probability that there are j busy trunks at some arbitrary time. The formulae involve products of integrals over the arrival distribution, but reduce to elementary functions if the arrival time distribution is also exponential. Since the overflow traffic will also have independent inter-arrival times with a distribution that can be determined from the formulas mentioned above, the author is able to also determine the distribution for the number of calls in the overflow trunks. The paper is written in a very clear and concise form.

G. Newell (Stockholm)

5531:

Kalecki, M. Mechanistic model of a random phenomenon. Zastos. Mat. 4 (1958), 113–129. (Polish. Russian and English summaries)

The author considers m watches, each with one hand and with a dial of unit circumference. On the kth watch the dial is divided into a white and a black arc, of lengths p_k and $1-p_k$, and the hand moves with velocity θ_k. Once every time-unit all watches are inspected and the number of hands pointing at black arcs is observed. Event "A" is recorded when that number is even, "B" when it is odd. Let $(A)_{Nm}$ and $(B)_{Nm}$ be the relative frequencies of A's and B's in N time-units; $(A_jA_{j+a})_{Nm}$ the relative frequency of the event: A occurs and after a time units A occurs; $(A_jB_{j+a})_{Nm}$ the relative frequency of the event: A occurs and after a time-units B occurs; etc. A number of theorems are proven; typical of these are the following: If $|1-2p_k|\leqq\alpha<1$ for all k, then

$$\lim_{m\to\infty}\lim_{N\to\infty}(A)_{Nm}=\lim_{m\to\infty}\lim_{N\to\infty}(B)_{Nm}=\tfrac{1}{2};$$

and, under additional assumptions on velocities θ_k,

$$\lim_{m\to\infty}\lim_{N\to\infty}(A_jA_{j+a})_{Nm}=\lim_{m\to\infty}\lim_{N\to\infty}(B_jB_{j+a})_{Nm}=$$
$$\lim_{m\to\infty}\lim_{N\to\infty}(A_jB_{j+a})_{Nm}=\lim_{m\to\infty}\lim_{N\to\infty}(B_jA_{j+a})_{Nm}=\tfrac{1}{4}.$$

Z. W. Birnbaum (Seattle, Wash.)

5532:

✱Kampé de Fériet, J. Fonctions harmoniques aléatoires dans le cercle-unité. Comptes-Rendus du 80ème Congrès des Sociétés Savantes, Lille, Juin 1955, pp. 411–415. Gauthier-Villars, Paris, 1955.

The author remarks that if, in the Poisson-Stieltjes integral for a positive harmonic function u in a disc, in terms of a measure on the bounding circle, the measure is randomized (that is, if the measure is made to depend on a parameter in a probability measure space), then $u(P)$ becomes a random variable at each point P of the disc. For a similar, but slightly more specialized result involving harmonic functions on a half-plane, see his earlier paper [C. R. Acad. Sci. Paris **237** (1953), 1632–1634; MR **15**, 449].

J. L. Doob (Urbana, Ill.)

5533:

Pyke, Ronald. On renewal processes related to type I and type II counter models. Ann. Math. Statist. 29 (1958), 737–754.

"Several renewal processes related to the type I and type II counter models are defined and studied. The distribution and characteristic functions for the secondary (or output) process of the type I counter model are obtained explicitly. Both the non-stationary and stationary probabilities of the state of the counter (locked or unlocked) are derived. Integral equations determining the distribution and characteristic functions for the secondary process of the type II counter model are obtained. Also it is shown that a more general model proposed by Albert and Nelson [same Ann. **24** (1953), 9–22; MR **14**, 775] may be solved explicitly in terms of a corresponding type II counter model. An example of this general model is given. Related with each model is a discrete renewal process which is also studied." (Author's summary)

W. L. Smith (Chapel Hill, N.C.)

5534:
Smith, Walter L. Renewal theory and its ramifications. J. Roy. Statist. Soc. Ser. B. 20 (1958), 243–302.

This paper is chiefly expository, with some new results and points of view. The author's summary follows.

"This is an expository article on the theoretical and some computational aspects of renewal theory. § 1 describes basic theory, including Blackwell's theorem; renewal density theorem; cumulants and asymptotic normality of the number of renewals in $(0, t)$; and the integral equations of renewal theory. § 2 is concerned with asymptotic behaviour of processes having an embedded renewal process: regenerative stochastic processes; semi-Markov processes; cumulative processes. § 3 discusses infinite sums and products connected with a renewal process which arise out of the study of electronic particle counters, and an integral equation connected with the infinite products. § 4 describes some generalizations of renewal theory that have been proposed by different writers and also 'infinitesimal' renewal processes. Illustrative examples are drawn from biology and from the theories of queues, dams, and electronic counters; a table summarizing some of the papers on the last subject is given."

T. E. Harris (Santa Monica, Calif.)

5535:
Leonov, Yu. P.; and Tel'ksnis, L. A. Estimation of parameters of the probability distribution of a random function with incomplete a priori information. Avtomat. i Telemeh. 18 (1957), 985–998. (Russian. English summary)

Let $y(t)=g(t)+n(t)$, where $\{n(t), -\infty<t<\infty\}$ is an ergodic process, with $E\{n(t)\}=0$, $E\{n(t)n(t+\tau)\}=R_n(\tau)=$ known function of τ. The function $g(t)$ is a non-random function about which no a priori information is available except that it is analytic. By modifying the procedure described by J. R. Ragazzini and this reviewer [J. Appl. Phys. 21 (1950), 645–655; MR 12, 347], the authors develop a trial and error technique for determining the impulsive response of a finite-memory filter which yields the "best" estimate (in least squares sense) of $g(t_0)$, t_0 being a fixed value of t. The estimate yielded by the optimal filter is not unbiased, and its determination requires, in general, operations involving both the ensemble and time averaging. The authors suggest various applications for their technique, among them the design of selfadjusting filters and the estimation of correlation functions.

L. A. Zadeh (New York, N.Y.)

5536:
Longuet-Higgins, M. S. The distribution of the sizes of images reflected in a random surface. Proc. Cambridge Philos. Soc. 55 (1959), 91–100.

The author calculates $p^*(I)$, the probability density of the image size I of a distant source (e.g., the sun) reflected in a gaussian random surface (e.g., the sea). It is shown that the distribution of I is proportional to the distribution of $|\Omega|^{-1}$ at specular points, where Ω is the total curvature of the surface. The distribution of Ω at points selected at random was derived in a previous paper by the author [Proc. Cambridge Philos. Soc. 54 (1958), 439–453; MR 20 #2043]; it depends on two parameters H and λ related to the energy spectrum of the surface, and involves a definite integral evaluated numerically by D. Catton and B. G. Millis [ibid., 454–462; MR 20 #2077]. Using these results, the author gives curves and formulas for $p^*(I)$, and discusses the dependence of $p^*(I)$ on H and λ.

R. A. Silverman (New York, N.Y.)

STATISTICS

See also 5513, 5519, 5535, 5692, 5701.

5537:
Smith, C. D. On the mathematics of simple correlation. Math. Mag. 32 (1958), 57–69.

Elementary expository article.

C. Davis (Providence, R.I.)

5538:
Jasper, Samuel J. A note on the standard deviation. Bul. Inst. Politehn. Iași (N.S.) 3 (1957), 39–42. (Russian and Romanian summaries)

5539:
Sarmanov, O. V. Maximum correlation coefficient (non-symmetrical case). Dokl. Akad. Nauk SSSR 121 (1958), 52–55. (Russian)

Let $F(x, y)$ be a joint density for random variables x and y. Let $p(x)$ and $P(y)$ be the marginal densities and $K(x, y)=F(x, y)/\sqrt{(p(x)P(y))}$. If $K(x, y)$ is symmetric the maximum correlation coefficient is defined as $R^*=1/\lambda_1$, where λ_1 is the smallest eigenvalue greater than 1 for the kernel K. The definition for unsymmetric kernels is defined in terms of symmetric kernels obtained from K in the usual manner. The author shows that $R^*=0$ if and only if x and y are independent and that in the case of linear regression R^* is the usual correlation coefficient.

J. L. Snell (Palo Alto, Calif.)

5540:
Wegmüller, Walter. Das Grenzverhalten statistischer Prüfverteilungen. Mitt. Verein. Schweiz. Versich.-Math. 58 (1958), 127–150.

This is an expository paper giving the basic distribution theory of the normal, chi-square, t and F distributions.

R. Pyke (New York, N.Y.)

5541:
Cox, D. R. The regression analysis of binary sequences. J. Roy. Statist. Soc. Ser. B. 20 (1958), 215–242.

This is an extensive survey of the subject in the title. Binary sequences are successions of independent observations, each of which may be either 0 or 1 (success or failure, etc.); for the regression these observations are taken as dependent either on preselected independent variables (one or two) or on independent variables which are functions of the sequence. Both tests and estimates of regression coefficients are considered, and a number of examples are considered in detail. A feature is the use of the logistic function to express the dependence: in the case of a single preselected independent variable the form is $\theta_i=\exp(\alpha+\beta x_i)/(1+\exp(\alpha+\beta x_i))$, with θ_i the probability that the ith observation is a 1, x_i the corresponding independent variable, and α, β the regression coefficients, α a nuisance parameter and β the item of main interest.

J. Riordan (New York, N.Y.)

5542:
Whittle, P. On the smoothing of probability density functions. J. Roy. Statist. Soc. Ser. B. 20 (1958), 334–343.

Let $x_1, x_2, \cdots, x_N$ be N independent observations from a population with probability density function $f(x)$. The author considers the problem of estimating $f(x)$ by linear functions $\hat{f}(x)=\sum_{i=1}^{N} w_x(x_i)$, where the weights $w_x(y)$ are to be found. He supposes that the density functions have a prior distribution with $E[f(x)]=\mu(x)/M$ and $E[f(x)f(y)]=\mu(x, y)/M^2$, say; the expectations being over the prior

distribution. The weights are then chosen to minimize the mean-square of $(\hat{f}(x)-f(x))$, where the mean is over the prior distribution and over the sampling distribution obtained by letting N be a Poisson variable with mean M. (The point of introducing the Poisson assumption is to simplify the equations, particularly (7)). The optimum weights are then the solutions of the integral equation

$$\mu(y)w_x(y)+\int (y,z)w_x(z)dz=\mu(y,x).$$

The weights are shown to be such that the estimates are invariant under a differentiable monotone transformation of x. The special case where $\mu(x,y)/[\mu(x)\mu(y)]^{\frac{1}{2}}$ depends only on $x-y$ is discussed, both for finite and infinite ranges of x. The mean squared deviations of $f(x)$ are shown not to tend to zero faster than M^{-1} as $M\to\infty$. Similar problems are considered by M. Rosenblatt [Ann. Math. Statist. **27** (1956), 832–837; MR **18**, 159].

D. V. Lindley (Cambridge, England)

5543:

Persson, Olle. On the solution of an over-determined system of equations by the method of least squares. Nordisk Mat. Tidskr. **6** (1958), 69–77, 95. (Swedish. English summary)

Expository. *C. C. Craig* (Ann Arbor, Mich.)

5544:

Constantine, A. G.; and James, A. T. On the general canonical correlation distribution. Ann. Math. Statist. **29** (1958), 1146–1166.

In part A of this paper there is an elementary derivation of Bartlett's [Ann. Math. Statistics **18** (1947), 1–17; MR **8**, 474] results on the distribution of the canonical correlation coefficients using exterior differential forms [A. T. James, Ann. Math. Statistics **25** (1954), 40–75; MR **15**, 726]. The method consists of taking the original multivariate normal distribution, transforming to the canonical correlations and other variables, and then integrating out these extraneous variables. In part B of this paper there is a new method of calculating the conditional moments which appear in Bartlett's expansion of this distribution, based on the process of averaging over the orthogonal group [A. T. James, Proc. Roy. Soc. London. Ser. A **229** (1955), 367–375; MR **17**, 53; H. Weyl, "The classical groups. Their invariants and representations." Princeton Univ. Press, 1939 (2nd ed., 1946); MR **1**, 42]. This method allows the calculation of moments of any order. An appendix contains the results calculated for various terms in certain of the conditional moments. *S. Kullback* (Washington, D.C.)

5545:

Watanabe, Yosikatsu; Isida, Makoto; Taga, Seiichi; Ichijô, Yosihiro; Kawase, Takaichi; Niside, Gôsuke; Takeda, Yosiharu; Horisuzi, Akira; and Kuriyama, Isamu. Some contributions to order statistics. J. Gakugei Tokushima Univ. **8** (1957), 41–90.

The main contributions of this paper consist of detailed analytical derivations of formulas for the joint distributions of order statistics and their moments for samples from a normal distribution, and in the derivation of the distribution of a linear function of order statistics for samples of size 3. The paper also contains a treatment, by the methods of maximum likelihood and least squares, of the problem of estimating the mean and standard deviation of a normal distribution from censored samples. {No references are given to papers published after 1950. Cf., e.g., Ruben, Biometrika **41** (1954), 200–227 [MR **16**, 153]; Teichroew, Ann. Math. Statist. **27** (1956), 410–426 [MR **18**, 238]; and Sarhan and Greenberg, ibid. **27** (1956), 427–451 [MR **18**, 238].}

D. M. Sandelius (Göteborg)

5546:

Sadowski, W. Statistical decision functions and the theory of games. Prace Mat. **2** (1958), 255–268. (Polish. Russian and English summaries)

A survey article.

5547:

Rao, M. M. Note on a remark of Wald. Amer. Math. Monthly **65** (1958), 277–278.

By the expressions $a_m=h_1+ms$, $r_m=h_2+ms$ given by Wald ["Sequential analysis", Wiley, New York, 1947; MR **8**, 593] for the acceptance and rejection values a_m and r_m of a lot, characterized by the probabilities p_0 and p_1 $(0<p_0<p_1<1)$, the value s defined by the equality

$$s\log\left(\frac{p_1}{p_0}\,\frac{1-p_0}{1-p_1}\right)=\log\frac{1-p_0}{1-p_1}$$

verifies the inequality $p_0<s<p_1$. This assertion of Wald in a note of his book is demonstrated by the author in a most simple elementary way. *O. Onicescu* (Bucarest)

5548:

Kamat, A. R. Contributions to the theory of statistics based on the first and second successive differences. Metron **19** (1958), no. 1–2, 97–118.

Let x_t $(t=1, 2, \cdots, n)$ be a random sample taken from a normal distribution with mean μ_t and constant variance σ^2. Define the sample variance: $s^2=\sum_{t=1}^{n}(x_t-\bar{x})^2/(n-1)$, where $\bar{x}$ is the sample mean. Also: $d=\sum_{t=1}^{n-1}|\Delta x_t|/(n-1)$, $d_2=\sum_{t=1}^{n-2}|\Delta^2 x_t|/(n-2)$, $\delta^2=\sum_{t=1}^{n-1}(\Delta x_t)^2/(n-1)$, $\delta_2^2=\sum_{t=1}^{n-2}(\Delta^2 x_t)^2/(n-2)$. Mean and variance of δ_2^2 and δ^2 are expressed in terms of $c_t=\Delta^2\mu_t/\sigma$. The bias of s^2, δ^2 and δ_2^2 is evaluated for a quadratic trend in μ_t. Examples: (1) 30 successive days of calorific value of a gas obtained in a colliery; (2) monthly consumers price index for Bombay City, 25 months beginning Jan. 1952.

Let $w^2=\delta^2/s^2$, $W=d/s$, $w_2^2=\delta_2^2/s^2$, $W_2=d_2/s$. Theorem: Let ϕ be a homogeneous function of degree k in the variables $x_t-\bar{x}$; the moments about zero of the ratio ϕ/s^k are given by: $\mu_r'=\mu_r'(\phi)/\mu_{rk}'(s)$. The first four moments about zero and the first four central moments are given for w_2^2 and W_2, also tables for μ_2, σ, β_1 and β_2 for $n=5$, 7, 10(5)30(10)50, 100. Covariances and correlation coefficients are computed and tables provided for the correlations between: δ^2 and s^2; d and s; δ_2^2 and s^2; d_2 and s; for $n=5(5)30(10)50$, 100, ∞.

The central limit theorem for dependent variables [W. Hoeffding and H. Robbins, Duke Math. J. **15** (1948), 773–780; MR **10**, 200] shows that the quantities δ^2, d, δ_2^2, d_2 are asymptotically normally distributed with means and asymptotic variances given. A theorem of H. Cramér ["Mathematical methods of statistics", Princeton Univ. Press, 1946; MR **8**, 39; pp. 254, 259] is used to prove the asymptotic normality of wW^2, w_2^2 and W_2, also for $u^2=\delta_2^2/\delta^2$ and $U=d_2/d$, whose means and asymptotic variances are given. The correlation coefficients for δ_2^2 and δ^2, and d_2 and d, are evaluated and tabulated for $n=5(5)30(10)50$, 100, ∞. There is a complete bibiography. *G. Tintner* (Lisbon)

5549:
Miyake, Saburo. An analysis of experimental data on extensive air showers. Progr. Theoret. Phys. 20 (1958), 844–856.

The author discusses in a qualitative fashion an interesting method of obtaining the cosmic ray shower curve (total number of particles as a function of atmospheric depth) without taking into account the fluctuations in the starting points of air showers. These fluctuations are important in the analysis of air shower data and can lead to a considerable modification of the primary energy spectrum obtained from such an analysis.
H. Messel (Sydney)

NUMERICAL METHODS

See also 5161, 5169, 5386, 5416, 5647.

5550:
✱Salzer, Herbert E.; and Kimbro, Genevieve M. Tables for bivariate osculatory interpolation over a Cartesian grid. Convair Division of General Dynamics Corporation, San Diego, Calif., 1958. 40 pp.

The paper discusses the calculation of polynomial expressions which agree with a function $f(x, y)$ and its two first partial derivatives at prescribed points in the plane. The choice of polynomial and the matching points must be made with care, to avoid singularity in the matrix equation for determining the coefficients. Explicit expressions are given for a two-point fit, but for more points numerical solution of the relevant linear equations is recommended. An interesting suggested use of these formulae is in the solution of hyperbolic equations by the method of characteristics.

Special attention is given to osculatory fitting at points on a Cartesian grid. Explicit equations are given for two to five points, together with the chosen configurations and tables of coefficients, to five decimals at intervals of 0.1 in the two interpolatory arguments. Some remainder terms are given and an example illustrates the use of the formulae and tables.
L. Fox (Teddington)

5551:
Laasonen, Pentti. On the iterative solution of the matrix equation $AX^2-I=0$. Math. Tables Aids Comput. 12 (1958), 109–116.

The author proves that if all proper values of A are real and positive, the iteration $X^{(0)}=kI$, $X^{(i+1)}=\frac{1}{2}[X^{(i)}+(AX^{(i)})^{-1}]$ converges quadratically, k being a positive scalar. The theorem is of interest in its own right, but unfortunately not for the reason given. In the introduction it is stated (correctly) that the problem $(\lambda A-B)x=0$ is of frequent occurrence, that if A is positive definite the problem is equivalent to $(\lambda I-C)y=0$ with $C=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. This is, of course, correct. But if A is positive definite it can be expressed in the form $A^{-1}=V^TV$, and one can take $C=VBV^T$. Moreover, if the matrices A and B are not symmetric it is suitable to take $C=A^{-1}B$ or $C=BA^{-1}$.

The discussion itself could be simplified by observing that the triangular matrices that arise are of the form $\alpha_0 I+\alpha_1 J+\cdots+\alpha_s J^s$, where $J^{s+1}=0$, and J is null except along the first superdiagonal where each element is unity.
A. S. Householder (Oak Ridge, Tenn.)

5552:
Parodi, Maurice. Sur une méthode de localisation des valeurs caractéristiques de certaines matrices. C. R. Acad. Sci. Paris 247 (1958), 571–573.

This paper deals with the characteristic values of a matrix P, of order $2n$, which has the form $\begin{pmatrix} b_{11}A & b_{12}A \\ b_{21}A & b_{22}A \end{pmatrix}$, where A is a square matrix of order n. The results, which appear as inequalities involving the elements of A and the characteristic values of B, locate a region which contains the characteristic values of P.
P. S. Dwyer (Ann Arbor, Mich.)

5553:
Huzino, Seiiti. On calculating eigenvalues by the gradient method. Mem. Fac. Sci. Kyusyu Univ. Ser. A. Math. 12 (1958), 30–39.

The author considers the problem of finding eigenvalues α so that the system of linear equations $Ax=\alpha Bx$ will have a non-zero solution x. An algorithm is given for finding the eigenvalues which is analogous to the method of conjugate gradients [M. Hestenes and E. Stiefel, J. Res. Nat. Bur. Standards 49 (1952), 409–436; MR 15, 651]. Under various assumptions, including the assumption that the eigenvalues are real, it is shown that the algorithm gives all of the eigenvalues. To properly apply the procedure it is necessary to have an upper bound for $|\alpha_1-\alpha_2|^{-1}$, where α_1 and α_2 are, respectively, the largest and the smallest eigenvalues. The author's discussion is entirely theoretical, no consideration being given as to the practical application of the procedure for actual numerical calculation.
D. M. Young, Jr. (Austin, Tex.)

5554:
Wolfe, J. M. A determinant formula for higher order approximation of roots. Math. Mag. 31 (1957/58), 197–199.

A determinant formula is presented for the approximation of the zeros of a function when the agreement of the approximating function with the original function is extended to derivatives of higher orders at the point of contact.
E. Frank (Chicago, Ill.)

5555:
Ghosh, P. K. Detection and evaluation of a certain type of complex roots by Graeffe's root-squaring method. Bull. Calcutta Math. Soc. 49 (1957), 43–46.

The author shows that when in the root-squaring process one gets the behaviour of multiple roots, one should examine the pattern (vertical and horizontal) of signs in different stages of the root-squared equation, unless it is shown in advance that the given equation has no complex roots, otherwise one may get just the modulus of a complex root masquerading as the root itself.
S. Levy (Philadelphia, Pa.)

5556:
Moran, P. A. P. Approximate relations between series and integrals. Math. Tables Aids Comput. 12 (1958), 34–37.

A review of several procedures for evaluating $I=\int_{-\infty}^{\infty} f(x)dx$ by the sum $S=\sum_{n=-\infty}^{\infty} hf(nh+\delta)$ (where h is the interval width and δ is a positive constant) is presented. A brief discussion of the method of evaluating an integral over a finite range by first transforming the interval into one over $(-\infty, \infty)$ is given. The influence of the size of h is included as is the mention of the possible gains to be achieved by combining estimates based on several values of h. A review of explicit results, including error bounds, is given when $f(x)=(2\pi)^{-\frac{1}{2}}\exp\{-\frac{1}{2}x^2\}$.
M. Muller (New York, N.Y.)

5557:

Derendyaev, I. M. Approximation by means of chords corresponding to Chebyshev interpolation nodes. Dokl. Akad. Nauk SSSR **120** (1958), 21–24. (Russian)

The difference $f(x)-f(x_0)-[f(x_1)-f(x_0)](x-x_0)/(x_1-x_0)$ in $(\underline{x}, \bar{x})$ does not exceed the upper bound for the second derivative f'' times $m=\max|(x-x_0)(x-x_1)|$. This m is minimized by taking x_0, x_1 to be the roots of the Čebyšev polynomial of degree 2 for $(\underline{x}, \bar{x})$. This method is applied to solve non-linear integral and differential equations by a sequence of approximations obtained from linear equations. For example, the solution $x(s)$ of $x(s)=\int_0^1 K(s, t, x(t))dt$ is the limit of the sequence $x_n(s)$ defined by

$$x_{n+1}(s)=\int_0^1 \bar{K}(s, t)[x_{n+1}(t)-z_{0n}(t)]dt+\int_0^1 K(s, t, z_{0n}(t))dt,$$

$\bar{K}(s, t)=[K(s, t, z_{1n})-K(s, t, z_{0n})]/(z_{1n}-z_{0n})$, where $z_{0n}(t)=x_n(t)-r_n/\sqrt{2}$, $z_{1n}(t)=x_n(t)+r_n/\sqrt{2}$, $r_n=\text{const.}\, a^{2^n-1}$, and a is determined from the bounds for K_x' and K_{x^2}''. Conditions of convergence and an estimation of the error are given. *G. G. Lorentz* (Syracuse, N.Y.)

5558:

Luke, Yudell L. The Padé table and the τ-method. J. Math. Phys. **37** (1958), 110–127.

The paper deals with the differential equation $x(ax+b)dE/dx=(Ax+B)E-Ax$, where a, b, A, B denote constants. According to Laguerre, solutions may be approximated by rational functions $E_n=\varphi_n/f_n$, φ_n, f_n denoting polynomials of degree n. The author analyzes the error $R_n=E-E_n$ by studying $\lim_{n\to\infty} R_n(x)$ for fixed x. By means of previously obtained results which exhibit E, f_n, φ_n as hypergeometric series, and through the use of asymptotic forms for these series, the author obtains asymptotic expressions for R_n, for large n. The detailed treatment distinguishes three cases, depending on the vanishing of the parameters a and b. Thus, for the case $a\neq 0$, $b=0$, there is obtained

$$R_n(z)\sim\frac{(-1)^{n+1}e^{-1/z}\Gamma(\lambda+1)\Gamma(\frac{1}{2})}{2^\lambda(2z)^{2n+1}\Gamma(2n+\lambda+1)n^{\frac{1}{2}}},$$

where $z=ax/B$, $\lambda=A/a$. A slightly different form of the basic differential equation (type II) leads to somewhat different expressions for the R_n. Finally, some numerical examples are appended which indicate that the asymptotic forms for the R_n can predict the error of the nth rational approximation E_n quite closely, even for rather small n. *M. Lotkin* (Wilmington, Mass.)

5559:

Urabe, Minoru. Numerical determination of periodic solution of nonlinear system. J. Sci. Hiroshima Univ. Ser. A **20** (1956/57), 125–148.

Let first $\dot{x}=X(x, t)$ be an analytic system, periodic in t with period ω, $\varphi(t, x_0)$ its general solution $(\varphi(0, x_0)=x_0)$; the equation for a periodic solution $\varphi(\omega/2, x_0)=\varphi(-\omega/2, x_0)$ is solved by Newton's method, taking advantage of the fact that the Jacobian matrix is the solution of the variational equations; in the case where X depends on a parameter λ and a periodic solution for $\lambda=\lambda_0$ is known, the method simultaneously yields the first approximation (terms linear in $\lambda-\lambda_0$) of the solution and the Newton corrections for the initial value x_0. For an autonomous system $\dot{x}=X(x)$ having a known approximately periodic solution $\varphi(t)$ with the approximate period ω_0, a moving orthogonal reference system is introduced, one of the axes of which remains always tangent to the path φ; writing down the system in the new coordinates, the Newton corrections for the period and the initial value of the periodic solution are computed. Similarly for autonomous systems depending on a parameter λ in both cases where for $\lambda=\lambda_0$ an isolated periodic solution or a continuum of such solutions is known. At first sight these methods appear to be quite laborious. *J. L. Massera* (Montevideo)

5560:

Urabe, Minoru. Periodic solution of Van der Pol's equation with damping coefficient $\lambda=0$ (0.2) 1.0. J. Sci. Hiroshima Univ. Ser. A **21** (1957/58), 193–207.

The author uses his method [see preceding review] to compute the limit cycle of $\ddot{x}-\lambda(1-x^2)\dot{x}+x=0$ for the values of λ indicated. The tabulated results show that the period varies from 6.283 to 6.687, the amplitude from 2 to 2.009 and the characteristic exponent from 0 to -1.07. *J. L. Massera* (Montevideo)

5561:

Greenspan, Donald. On the numerical solution of n-dimensional boundary value problems associated with Poisson's equation. J. Franklin Inst. **266** (1958), 365–371.

Let R be an open, bounded simply connected region in n-dimensional Euclidean space with boundary S such that the boundary value problem

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}=f(x_1, x_2, \cdots, x_n) \text{ in } R,$$

$$u(x_1, x_2, \cdots, x_n)=g(x_1, x_2, \cdots, x_n) \text{ on } S$$

possesses a solution of class C^4 in $G+S$. Each $\partial^2 u/\partial x_i^2$ is replaced by the corresponding second central difference quotient with mesh length h_i, and the prescribed boundary values are transferred to the nearest interior grid points. The resulting finite difference problem is analyzed with respect to solubility and truncation error. This is achieved by a straightforward application of the method of Gerschgorin [Z. Angew. Math. Mech. **10** (1930), 373–382]. *W. Wasow* (Madison, Wis.)

5562:

Lotkin, Mark. The numerical integration of heat conduction equations. J. Math. Phys. **37** (1958), 178–187.

This paper is concerned with difference equation approximations to the equations of unsteady one-dimensional heat conduction through composite media:

$$\phi^{(m)}\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k^{(m)}\frac{\partial u}{\partial x}\right), \tag{1}$$

for $a_{m-1}\leq x\leq a_m$, $m=1, 2$, $a_0=0$, $0\leq t\leq T$, under the boundary conditions

$$-(k^{(1)}\partial u/\partial x)=g(t) \text{ for } x=0,\ t>0; \tag{2}$$
$$\partial u/\partial x=0 \text{ for } x=a_2,\ t>0;$$

the initial condition

$$u(x, 0)=f(x)\quad a_0\leq x\leq a_2; \tag{3}$$

and the condition

$$(k^{(1)}\partial^- u/\partial x)_{a_1}=(k^{(2)}\partial^+ u/\partial x)_{a_1}. \tag{4}$$

In (1), $k^{(m)}$ and $\phi^{(m)}$ denote known functions of u. The convergence of the approximations is established for $k^{(m)}$ constant, and the rate of convergence is estimated. In the last section, an example is given along with numerical data comparing the approximate with the exact solutions. *J. Elliott* (New York, N.Y.)

5563:

Forsythe, George E. Reprint of a note on rounding-off errors. SIAM Rev. 1 (1959), 66–67.

This is a reprint of "Note on rounding-off errors", National Bureau of Standards, Los Angeles, Calif., 1950 [MR 12, 208].

5564:

Bonfiglioli, Luisa. Nomogramma per la funzione $z=xy(1-x^2-y^2)$. Riveon Lematematika 12 (1958), 13–18. (Hebrew. Italian summary)

5565:

Valat, Jean. Détermination des solutions périodiques des équations différentielles non linéaires, dont les coefficients varient suivant une fonction créneau. C. R. Acad. Sci. Paris 247 (1958), 1961–1964.

This paper considers periodic solutions of the equation

$$y''+f_1(t)y+f_2(t)y^3=0,$$

where $f_r(t)$ $(r=1, 2)$ is of the form $a_{r1}+a_{r2}S(t)$, and $S(t)$ is the "crenellated" function taking the values $+1, -1, +1, -1, \cdots$ with a saltus each half period, $\frac{1}{2}T$. Three methods of solution are considered: (i) Construction of separate phase-plane diagrams for each half period (one as a transparent tracing), with time (t) graduations marked on the trajectories; by superposing the diagrams it is possible (having due regard to the t-graduations) to select closed trajectories representing harmonic and subharmonic solutions. (ii) Solution on an electronic analogue computer. (iii) Matching the formal solutions in terms of elliptic functions for each half period.

The author states that he proposes to extend his work to the study of boundaries between stable and unstable periodic solutions. *J. G. L. Michel* (Teddington)

COMPUTING MACHINES

See also 5142, 5565.

5566:

Polozova, N. G. La multiplication des séries trigonométriques à l'aide de la calculatrice électronique. Byull. Inst. Teoret. Astr. 6 (1958), 757–769. (Russian. French summary)

Il s'agit ici de la multiplication des séries trigonométriques. On obtient les formules nécessaires et c'est pour la machine BESM qu'on propose le programme de la solution du problème en question. *Résumé de l'auteur*

MECHANICS OF PARTICLES AND SYSTEMS

See also 5334, 5673, 5676, 5677, 5683, 5719.

5567:

Maiellaro, Michele. Sul campo delle accelerazioni nel moto di una figura rigida piana nel suo piano. Giorn. Mat. Battaglini (5) 5 (85) (1957), 264–270.

In plane kinematics the locus of the points for which there exists a given linear relation between the normal and the tangential acceleration is a conchoid. Special cases. *O. Bottema* (Delft)

5568:

Manarini, Mario. Sul campo delle accelerazioni nel moto di una figura piana nel suo piano. Giorn. Mat. Battaglini (5) 5 (85) (1957), 133–140.

Determination of the instantaneous center of acceleration in plane kinematics by means of vectorial considerations. *O. Bottema* (Delft)

5569:

Montaldo, Oscar. Sui sistemi autonomi che godono delle proprietà di Kasner. Rend. Sem. Fac. Sci. Univ. Cagliari 27 (1957), 107–113.

Kasner gave a characteristic set of four properties of the ∞^5 dynamical trajectories of a positional field of force in Euclidean space of three dimensions. See Edward Kasner, Differential-geometric aspects of dynamics, Amer. Math. Soc. Colloq. Publ., New York, 1913, 1934. The author extends these four properties to an arbitrary directional field of force in which the components are of the form: $\phi(x, y, z, \dot{x}, \dot{y}, \dot{z})$, $\psi(x, y, z, \dot{x}, \dot{y}, \dot{z})$, $\chi(x, y, z, \dot{x}, \dot{y}, \dot{z})$, where ϕ, ψ, χ, are homogeneous polynomials of degree $n\neq2$, in $(\dot{x}, \dot{y}, \dot{z})$ with coefficients functions of (x, y, z). *J. De Cicco* (Chicago, Ill.)

5570:

Benney, D. J. On the limiting equilibrium of n masses. Amer. Math. Monthly 65 (1958), 9–17.

A rigid framework rests on a rough horizontal plane in n points A_i. A horizontal force is applied at some point in a given direction and gradually increased until equilibrium is about to be disturbed; the instantaneous rotation center is I. The author asks whether I coincides with A_r; that means that there is a limiting friction force at each point A_s, $s\neq r$. Two special cases are considered: a light rod with equidistant points of equal mass; a regular polygon with equal weights at its vertices. *O. Bottema* (Delft)

5571:

Broman, Arne. A mechanical problem by H. Whitney. Nordisk Mat. Tidskr. 6 (1958), 78–82, 95–96.

A carriage moves for $0\leqq t\leqq T$ in a prescribed way on a horizontal rail; a rod on it can move around a horizontal axis through the lower end of the rod, its inclination being φ, $\varphi(0)=\alpha$, $\varphi(T)=\beta$. Courant and Robbins [What is mathematics, Oxford Univ. Press, New York, 1946; MR 3, 144; pp. 319–321] ask whether α may be chosen such that $\beta=\pi/2$, and they answer yes, assuming that β is a continuous function of α. The author digs deeper in the problem, discussing the differential equation for the motion of the rod. Assuming the acceleration of the carriage to be continuous the answer is still positive. One can also choose α so that the rod never falls down. *O. Bottema* (Delft)

5572a:

Novoselov, V. S. The application of Helmholtz's method to the study of motion of Chaplygin's system. Vestnik Leningrad. Univ. 13 (1958), no. 13, 102–111. (Russian. English summary)

5572b:

Novoselov, V. S. Application of the Helmholtz method to the motion of non-holonomic systems. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. 13 (1958), no. 1, 80–87. (Russian. English summary)

It was shown by Helmholtz that under certain conditions the equations of motion of a nonholonomic system can be written in Lagrangian form [see E. T. Whittaker, "Analytical dynamics", 4th ed., University

Press, Cambridge, 1937; Dover, New York, 1944; MR 6, 74; pp. 44–45]. In the first paper the author considers techniques for explicitly evaluating the generalized kinetic potential for such equations; two illustrative examples are worked out. In the second paper the procedures are studied for the special case of restraints of a type studied by Chapligin; it is shown how equations not satisfying the Helmholtz conditions can be written in a modified form so as to satisfy these conditions. The method is illustrated by the analysis of the rolling of a sphere on a rough horizontal plane.

W. Kaplan (Ann Arbor, Mich.)

5573:

Colombo, Giuseppe. Sopra il problema del "lacet". J. Math. Mech. 7 (1958), 483–501.

This is a contribution to the theory of the stability of uniform motion of a wheeled vehicle on a pair of rails. The departure from uniform motion is considered to be chiefly a combination of a sideways motion of the wheels on the rails and a pivoting of the vehicle about a vertical axis. The principal factors influencing such a motion are assumed to be the elasticity of the roadbed and the shapes of the cross-sections of the rails and wheels. The author derives a system of differential equations governing the motion, and draws a few simple conclusions concerning boundedness and stability of solutions. However, in view of the complexity of the equations and the large number of parameters appearing in them, he considers that a thorough study of the problem would require extensive work with a digital computer.

L. A. MacColl (New York, N.Y.)

5574:

Mack, C. Theory of the spinning balloon. Quart. J. Mech. Appl. Math. 11 (1958), 196–207.

The title describes the appearance of the envelope, but not the problem actually treated in this paper, which is that of yarn whirling at very high angular velocities, as occurs in textile operations. The dynamical equations are set up on the assumptions that the gravity effect can be ignored, and that the air drag effect is small, but not negligible. The steady state equations are solved for the case of zero air drag, and more generally, an expansion is given involving parameters dependent on air drag.

D. G. Bourgin (Urbana, Ill.)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 5607.

5575:

✱Collins, Frank C.; and Raffel, Helen. Transport processes in liquids. Advances in chemical physics, Vol. I, edited by I. Prigogine, pp. 135–164. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London; 1958. xi+414 pp. $11.50.

This article is well suited to give the non-specialist a superficial but clear insight into the existing theories of viscosity, thermal conduction and diffusion in liquids. A useful bibliography is appended, but there are inevitably some important omissions from this and the text.

H. S. Green (Adelaide)

5576:

Prigogine, I.; and Résibois, P. On the approach to equilibrium of a quantum gas. Physica 24 (1958), 795–816.

The so-called Uehling-Uhlenbeck equation describes the decrease of the number of particles n_a in state a due to collisions; for instance for a Bose gas:

$$-dn_a/dt=\sum_{bkl}\omega_{ab}^{kl}[n_an_b(1+n_k)(1+n_l)-n_kn_l(1+n_a)(1+n_b)].$$

The coefficients $\omega_{ab}{}^{kl}$ are usually computed from the two-body collision process. Here the authors start from the Schrödinger equation of the total n-body system, thus taking into account the effect of Bose statistics in the intermediate states. The gas is supposed sufficiently dilute so that intermolecular forces may be neglected in equilibrium, although they are of course essential for the $\omega_{ab}{}^{kl}$. The technique of Van Hove is used to obtain a 'master equation' for the diagonal terms of the density matrix. Thus two-particle collisions are treated to all orders in the perturbation, but collisions of three or more particles are neglected. The transition probabilities in this master equation are then expanded to second order in the occupation numbers n, and the assumption $\overline{n_an_b}=\bar{n}_a\bar{n}_b$ is made. In this way the above Uehling-Uhlenbeck equation is obtained, supplemented by new terms due to the statistics in intermediate states. These terms are also of the third power in the n, but their order of magnitude differs from the Uehling-Uhlenbeck terms by the factor (range of the force/de Broglie wave length).

N. G. van Kampen (Utrecht)

ELASTICITY, PLASTICITY

See also 5346, 5365.

5577:

Brown, Edmund H. On the most general form of the compatibility equations and the conditions of integrability of strain rate and strain. J. Res. Nat. Bur. Standards 59 (1957), 421–426.

The author claims to show that "recent statements on the significance of the compatibility equations, essentially that their satisfaction is equivalent to the condition that the space be locally Euclidean, are misinterpretations based on restricted forms of the equations." It is not made clear to which statements the author refers. He derives various formulae involving a rate of strain tensor. A common form of compatibility equations is that the Riemann tensor based on a suitably chosen positive definite strain tensor, regarded as metric tensor, vanishes. One possibility is the author's tensor g_{ij}. This is necessary and sufficient that the tensor be the metric tensor of a locally Euclidean space. A proof is given by L. P. Eisenhart [An introduction to differential geometry, Princeton Univ. Press, 1940; MR 2, 154; theorem 23.3].

J. L. Ericksen (Baltimore, Md.)

5578:

Ericksen, J. L.; and Truesdell, C. Exact theory of stress and strain in rods and shells. Arch. Rational Mech. Anal. 1 (1958), 295–323.

The authors provide a general description of stress and strain in rods and shells which is divorced from any constitutive assumptions intended to describe the elastic or plastic response of the material. They first give a sketch of those properties of an oriented body which are independent of the number of dimensions. Thereafter follow theories of finite strain of rods and shells: 1. Strain of position, which is separated into (1a) intrinsic theory and (1b) imbedding theory; and 2. Strain of orientation, which

is divided into two parts: (2a) differential description of the undeformed oriented body, and (2b) differential description of the deformation. Proofs of invariance and completeness are given. The paper closes with an exact description of stress in rods and shells. The problem of a satisfactory connection between stress and strain in rods and shells still remains, even for elastic materials.
A. E. Green (Newcastle-upon-Tyne)

5579:

Mazzarella, Franco. Le equazioni di congruenza in coordinate curvilinee. Rend. Accad. Sci. Fis. Mat. Napoli (4) 24 (1957), 59–64.

Some equations from the theory of elasticity are transformed from a Cartesian frame to a system of orthogonal curvilinear coordinates. *O. Bottema* (Delft)

5580:

Gurevič, G. I. The relationship between the stress tensor and deformation rate tensor in the general case of large and small deformations. Dokl. Akad. Nauk SSSR 120 (1958), 987–990. (Russian)

[A translation of this paper appeared in Soviet Physics. Dokl. 3 (1958), 672–675.] The central result of this paper is a derivation of the stress-deformation relation

$$(1)\quad 2\mu T(\sigma,\theta)d_{ij}=\sigma_{ij}-\delta_{ij}\sigma_{av}+T\frac{d}{dt}\left[\sigma_{ij}-\delta_{ij}\frac{3\nu}{1+\nu}\sigma_{av}\right],$$

where $d_{ij}=\frac{1}{2}(v_{i,j}+v_{j,i})$ is the rate of deformation tensor, σ_{ij} is the stress, $\sigma_{av}\equiv\frac{1}{3}\sigma_{kk}$, and $T(\sigma,\theta)$ is a certain function of the stress and temperature whose explicit form is given by the author. The derivation of (1) is based on the idea of a variable reference state relative to which strain is measured and is assumed related to the stress by the usual linear isotropic stress-strain relations of elasticity theory. The reference state deforms or relaxes in accordance with a formula derived by the author in a previous paper [Trudy Geofiz. Inst. Akad. Nauk SSSR **1953**, no. 21(148), 49]. The stress-deformation relations (1) have the general form

$$(2)\quad \frac{d\sigma_{ij}}{dt}=f_{ij}(\mathbf{d},\sigma)$$

so that, except for some missing non-linear terms on the left-hand side of (2) needed to make the time derivative of the stress properly invariant under rigid motions, a material described by (1) is "hypo-elastic", as defined by Truesdell [J. Rational Mech. Anal. 4 (1955), 83–133; MR **16**, 880]. *R. A. Toupin* (Washington, D.C.)

5581:

Wigglesworth, L. A. Stress relief in a cracked plate. Mathematika 5 (1958), 67–81.

This paper determines the distribution of stress in an infinite elastic plate in which there is a straight crack of finite length; one end of the crack is terminated by a circular hole with centre lying on the continuation of the line of the crack, so that the theory is relevant to the practice of preventing the spreading of a line crack by drilling holes at its ends. An exact solution is given for a uniform stress at infinity, in the conditions of either plane stress or plane strain. The solution of the integral equation arising from the unknown stresses along the time of the crack is obtained by Mellin transforms in an extension of the method worked out by the author in a recent paper [Mathematika 4 (1957), 76–96; MR **19**, 995].
W. R. Dean (London)

5582:

Atsumi, Akira. Stress concentrations in a strip under tension and containing an infinite row of semicircular notches. Quart. J. Mech. Appl. Math. 11 (1958), 478–490.

Continuing his previous study [J. Appl. Mech. 24 (1957), 565–573; MR **19**, 998] on semi-circular notches in a strip under tension, the author now treats the case of an infinite number of symmetrically placed semi-circular notches. He constructs two biharmonic stress functions in polar coordinates and then adopts Ling's [ibid. 14 (1947), A275–A280; MR 9, 256] perturbation technique to solve the infinite set of linear equations in the unknown constants. A number of numerical results and graphs are given to study the decrease in the concentration factor.
B. R. Seth (Kharagpur)

5583:

Hetényi, M.; and McDonald, P. H., Jr. Contact stresses under combined pressure and twist. J. Appl. Mech. 25 (1958), 396–401.

The paper is concerned with the displacements and stresses induced in an elastic half-space when an elastic sphere is pressed against it and is twisted until complete slip occurs. Let (r, θ, z) be circular cylindrical coordinates, the z-axis being coincident with the axis of symmetry and the plane $z=0$ being the boundary of the semi-infinite medium. Following Coulomb's dry-friction hypothesis, the authors assume the distribution of the shearing tractions over the contact region ($0\leq r\leq a$, $z=0$) in the form $\tau_{\theta z}=fp(r)$, where $p(r)=p_0(1-r^2/a^2)^{\frac{1}{2}}$ is the Hertzian pressure distribution and f is the coefficient of friction. Since the Hertzian displacement and stress fields are known, the present problem reduces to one of pure torsion. This torsion problem is solved in terms of definite integrals involving Bessel functions, which are evaluated by numerical integration. Numerical results are given for the surface displacements and stresses, as well as for the values of the maximum shear stress at several levels below the boundary. These values, along with the surface values of $\tau_{\theta z}$, are found to be in good agreement with the results of photoelastic measurements which, in particular, confirm the validity of Coulomb's hypothesis in the present instance.
E. Sternberg (Providence, R.I.)

5584:

Kil'čevs'kiĭ, M. O. Extremal properties of the solution of the compression contact problem of elastic solids. Dopovidi Akad. Nauk Ukraïn. RSR 1958, 17–20. (Ukrainian. Russian and English summaries)

The dynamical contact problem of two elastic bodies with regular surface is considered. Since the equation of motion of the center of gravity and the basic kinematical equation of Hertz do not determine the problem completely, the variational equation of Gauss is adopted involving contact pressure and area, density of the bodies and thickness of their molecular boundary layer.
J. Nowinski (Madison, Wis.)

5585:

Krug, E. M. On the theory of thick rectangular slabs. Černivec. Derž. Univ. Nauk. Zap. Ser. Fiz.-Mat. Nauk Kiïv 4 (1952), no. 2, 3–38. (Russian)

5586:

Amenzade, Yu. A. Regularity of an infinite system of equations in the bending problem of a circular prismatic beam with elliptic cavity. Dokl. Akad. Nauk Azerbaĭdžan. SSR 11 (1955), 155–160. (Russian. Azerbaĭdžani summary)

5587:

Hu, Hai-chang. On the theory of uniformly loaded anisotropic cantilever beams. Sci. Sinica 6 (1957), 21–31.

5588:

Dutt, S. B. Stress concentrations around a small spherically isotropic spherical inclusion on the axis of an isotropic circular cylinder in torsion. J. Tech. Bengal Engrg. Coll. 3 (1958), 13–17.

Extension of Das's solution [J. Appl. Mech. 21 (1954), 83–87] to the case of a spherically isotropic spherical inclusion. Some minor errors in the references are observed. *S. C. Das* (Chandernagore)

5589:

Kaliski, Sylwester. The dynamical problem of the rectangular parallelepiped. Arch. Mech. Stos. 10 (1958), 329–370. (Polish and Russian summaries)

The paper considers the forced vibrations of a rectangular parallelepiped with all six walls fixed under the action of a mass force field of frequency ω. It is first shown how the problem can be reduced to the solution of an infinite system of algebraic equations. A proof is then given of the existence and uniqueness of the solution for all values of the forcing frequency, ω, between zero and the fundamental frequency of the parallelepiped. This proof can be extended to other values of ω, with the restriction that the free vibration frequencies of the system be avoided.

In principle, the solution can be evaluated by direct application of an iterative scheme involving finite systems of equations. The convergence of such schemes emerges as a by-product of the existence and uniqueness proof. Since such an approach would be tedious in actual application however, the author outlines an alternate method involving solution of a simpler (infinite) multiplicative system of equations which approximates the original one. Bounds on the error resulting from the approximation can be obtained quite readily. The method has the advantage that an increase in the number of unknowns determined is not accompanied by an excessive increase in labor. The paper is limited to a general theoretical discussion; no numerical example is given.

It is noted by the author that if the walls of the parallelepiped are stress free rather than fixed, the problem is entirely analogous and can be handled in a similar manner. *P. Mann-Nachbar* (Palo Alto, Calif.)

5590:

Fulton, J. An integral transform solution of the differential equation for the transverse motion of an elastic beam. Proc. Edinburgh Math. Soc. 11 (1958/59), 87–93.

The author develops a finite integral transform suitable for the resolution of boundary value problems related to the fourth order partial differential equation governing the forced transverse motion of an elastic beam. The inversion theorem for this transform is in the form of a series expansion of the characteristic functions of the homogeneous differential equation. The method of solution is illustrated by considering the motion of a beam (*a*) of uniform cross-section, (*b*) in the form of a truncated cone with circular cross-section. *I. N. Sneddon* (Glasgow)

5591:

Samuels, J. C.; and Eringen, A. C. Response of a simply supported Timoshenko beam to a purely random Gaussian process. J. Appl. Mech. 25 (1958), 496–500.

In a previous publication A. C. Eringen [J. Appl. Mech. 24 (1957), 46–52; MR 19, 340] calculated the eigenfunction series for the displacement $w(x, t)$ of a beam subject to an applied load $P(x, t)$ on the basis of the linearized Bernoulli-Euler theory. From it he calculated the autocorrelation function of w on the assumption that the correlation function of $P(x, t)$ with $P(\xi, t-\tau)$ is $\delta(x-\xi)\delta(\tau)$. The corresponding series for the autocorrelation function of the stresses diverged. In the present paper the calculation is repeated on the basis of the linearized Timoshenko theory with the inclusion of a fictitious term due to damping of rotatory motion. A finite result is obtained for the autocorrelation function of the stresses.

The fact that the original divergence resulted from the use of a load with a singular autocorrelation function makes the original difficulty appear to be spurious. The fact that translatory velocity damping and internal damping did not eliminate it supports this view. *J. B. Keller* (New York, N.Y.)

5592:

Gel'činskiĭ, B. Ya. Reflection and refraction of an elastic wave of arbitrary shape by a curved interface between two media. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 458–460. (Russian)

The author considers briefly the reflection and refraction of an elastic wave front of arbitrary shape incident upon a boundary of arbitrary shape separating two solids with different mechanical properties. By using the ray method for elastic waves [V. M. Babič, Dokl. Akad. Nauk SSSR 110 (1956), 355–357; MR 19, 804] the author claims to prove the previously postulated "principle of the isolated element" — that the incident elastic wave is reflected at a point on the curvilinear boundary as if it were a plane wave falling on a small element of the surface situated at that point. *I. N. Sneddon* (Glasgow)

5593:

Chakraborty, Sakti Kanta. Elasto-dynamic problem concerning a centre of rotation in a semi-infinite medium of transversely isotropic material. Proc. Nat. Inst. Sci. India. Part A 24 (1958), 250–255.

The author writes down a simple dynamical solution of the equations governing the motion of an elastic solid which has transverse isotropy. This solution corresponds to a displacement field of such a nature that, in cylindrical coordinates r, θ, z, only the component in the θ-direction is different from zero. This has the simplifying effect that only one wave velocity is involved. The effect of such a disturbance in a semi-infinite elastic solid is also discussed, and some particular cases (of, however, very little physical interest) are reported. *I. N. Sneddon* (Glasgow)

5594:

Babič, V. M. The Sobolev-Kirchhoff method in the dynamics of non-homogeneous elastic solids. Vestnik Leningrad. Univ. 12 (1957), no. 13, 146–160. (Russian. English summary)

Using Schwartz's theory of distributions, the author extends the Sobolev solution of the Cauchy problem [Math. Sb. 1 (1936), 39–71; see p. 43] to cover the case of the propagation of waves in an infinite inhomogeneous elastic solid. The analysis is carried out for general initial values of the displacement vector and its time derivative; no specific problems are solved to illustrate the use of the method. *I. N. Sneddon* (Glasgow)

5595:

Knopoff, Leon; and MacDonald, Gordon J. F. Attenuation of small amplitude stress waves in solids. Rev. Mod. Phys. 30 (1958), 1178–1192.

This paper first reviews the theory of the propagation of stress waves in a medium which is linearly viscoelastic. Expressions for the specific dissipation function ($1/Q$) in Maxwell and Voigt solids are derived, and the behavior of the more general Boltzmann solid is briefly considered. Next follows a review of experimental results from seismology and acoustics; these indicate that, for a wide variety of materials, the dissipation function is approximately independent of frequency. Finally, theories of non-linear permanent deformation are discussed in terms of Eckart's thermodynamic approach to irreversible processes and, in particular, the behavior of a model involving a non-linear hysteresis loop is treated in detail. The possibility of a Coulomb type of internal friction which would lead to a frequency independent dissipation is also discussed. *H. Kolsky* (London)

5596:

Arutyunyan, N. H.; and Manukyan, M. M. Creep of composite cylindrical pipes. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk 10 (1957), no. 6, 41–58. (Russian. Armenian summary)

5597:

Arutyunyan, N. H.; and Čobanyan, K. S. Bending of prismatic rods composed of various materials, with account taken of creep. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk 10 (1957), no. 5, 59–72. (Russian. Armenian summary)

5598:

Bland, D. R.; and Naghdi, P. M. A compressible elastic, perfectly plastic wedge. J. Appl. Mech. 25 (1958), 239–242.

An infinitely long straight-sided acute wedge is loaded with a uniform normal pressure on one side and is stress free on the other. A state of plane strain is assumed. The wedge material is elastic-perfectly plastic. The elastic strains are compressible and satisfy Hooke's law; the plastic strain rates are incompressible and satisfy Tresca's yield condition and the associated flow rule.

The solution is found to depend upon a single space coordinate representing an angle at the vertex of the wedge and upon the angle of the elastic-plastic boundary. Complete solutions are obtained for the stresses and displacements. Graphical comparisons with an elastically incompressible material are given for the particular case of a wedge of 45°. *P. G. Hodge, Jr.* (Chicago, Ill.)

5599:

Išlins'kiĭ, O. Yu. Extension of an infinitely long ideally plastic bar of variable cross-section. Dopovidi Akad. Nauk Ukraïn. RSR 1958, 12–16. (Ukrainian. Russian and English summaries)

An infinitely long bar, made of perfectly-plastic material, is subject to axial tension. The bar is not assumed to be of uniform cross-section. The problem posed is the determination of the shape of the bar for the existence of a continuous plastic state. Under the assumed two-dimensional conditions, the problem is statically-determinate. Specific results, somewhat restricted in nature, are found. However, no attention is given to the determination of compatible velocity fields.

H. G. Hopkins (Fort Halstead)

5600:

Owen, M. J. An elastoplastic analysis of a rotating annulus. J. Franklin Inst. 267 (1959), 55–68.

If account is taken of the change in thickness due to straining of an initially flat rotating annulus, the defining equations become nonlinear. However, if all quantities of interest are expanded in a power series in $\mu=K/G$, a series of linear problems is obtained. Here K is the yield stress in pure shear and G the elastic shear modulus. For most structural materials, μ will be between 0.001 and 0.01.

The author considers an elastic-perfectly plastic material which satisfies Tresca's yield condition and the associated flow rule in the plastic range. The stress and displacement solution for the title problem are found in terms of the radius r and elastic-plastic boundary ρ. Explicit solutions are given for the first non-zero term in the expansion of stress, displacement, and speed. Numerical results are then given for the first correction term. The correction for fully plastic speed is found to be less than 4% in the worst case considered, and generally less than 1%. Similar results are found for stresses and displacements, except that for very small bore holes the strain corrections are greater.

P. G. Hodge, Jr. (Chicago, Ill.)

5601:

Zadoyan, M. A. Variational equations of the theory of creep. Akad. Nauk Armyan. SSR. Dokl. 26 (1958), 263–268. (Russian. Armenian summary)

This is an extension to creep phenomena of the variational principles of Lagrange, Castigliano and Reissner, as well as of the formula of Clapeyron.

J. Nowinski (Madison, Wis.)

5602:

Poritsky, H.; and Fend, F. A. Relief of thermal stresses through creep. J. Appl. Mech. 25 (1958), 589–597.

A general nonlinear system of partial differential equations is derived involving (1) Hooke's law, for elastic strain, (2) experimentally determined relations between the rate of creep, stress and plastic strain, (3) total strain-displacement relations, and (4) differential equations of stress equilibrium. To reduce the problem to a tractable form, the variation of the elastic constants and the coefficient of thermal expansion with temperature are neglected. It is assumed that during each time interval the stresses and also the rates of creep remain constant. On the basis of the Huber-Mises-Hencky hypothesis, the tensile test stress-strain creep relations are generalized to the corresponding relations for the compound states of stress. The axially symmetrical case involving a quickly heated infinitely long cylinder with parabolic temperature distribution is treated in detail by solving repeatedly a second order differential equation for radial displacement. Gradual relief of stress caused by creep is illustrated by curves.

J. Nowinski (Madison, Wis.)

5603:

Yüksel, Halil. Elastic, plastic stresses in free plate with periodically varying surface temperature. J. Appl. Mech. 25 (1958), 603–606.

The one-dimensional problem of the elastic-plastic distribution of stresses in a free plate arising from a time-dependent temperature gradient produced by a periodically varying surface temperature on one face and a constant temperature on the other is solved under the

assumption of ideal elastic-plastic behavior and symmetry of the stress distribution about the center-plane of the plate. *A. M. Freudenthal* (New York, N.Y.)

5604:

Brull, Maurice A.; and Vinson, Jack R. Approximate three-dimensional solutions for transient temperature distribution in shells of revolution. J. Aero/Space Sci. **25** (1958), 742–750.

An approximate three-dimensional heat conduction equation is formulated for thin shells of revolution by making assumptions similar to those of the theory of thin elastic shells. It is then shown that the resulting partial differential equation is separable for all shell shapes and that solutions may be obtained in terms of well-known functions. (From the authors' summary)

E. H. Mansfield (Farnborough)

5605:

Zadoyan, M. A. Thermal stress state of concrete blocks with account taken of the creep of the material. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **10** (1957), no. 5, 73–98. (Russian. Armenian summary)

The thermal stress in a concrete block resting on a plane foundation is investigated, the block having the form of a long rectangular strip or circular cylinder (its axis being perpendicular to the plane of the foundation). A deformational postulate generalizing the Bernoulli hypothesis of cross-sections remaining plane is assumed. The tangential friction forces between the block and the foundation are assumed to be proportional to the horizontal displacement. The thermoelastic problem, as well as the thermal problem involving creep (the latter following the theory of creep of Arutyunyan and for time dependent temperature field), are discussed assuming temperature variation along the thickness of the block.

J. Nowinski (Madison, Wis.)

STRUCTURE OF MATTER

5606:

Quinn, John J.; and Ferrell, Richard A. Electron self-energy approach to correlation in a degenerate electron gas. Phys. Rev. (2) **112** (1958), 812–827.

The correlation energy of a degenerate electron gas is calculated by a method which singles out one individual electron at the surface of the Fermi sea for inspection. The polarization of the gas around this electron is calculated, the action of which back on the electron is the self energy. This self energy is related to the derivative of the correlation energy with respect to density, and hence the latter can be obtained by an integration over density. The self energy is calculated by means of Feynman propagators which have to be altered slightly to include exchange effects. The "polarization propagator" is also derived from the momentum-exciton model. The self energy defines an optical potential whose negative imaginary part damps the one-electron state for momentum states appreciably far removed from the Fermi sea. For real metals the correlation energy is exactly equal to the self energy of the Fermi surface and hence no integration over density is necessary.

M. J. Moravcsik (Livermore, Calif.)

5607:

Garcia-Moliner, F. A variational calculation of electronic transport in a magnetic field. Proc. Roy. Soc. London. Ser. A **249** (1959), 73–89.

The work of an earlier paper [F. Garcia Moliner and S. Simons, Proc. Cambridge Philos. Soc. **53** (1957), 848–855; MR **19**, 1102] is extended to a study of the transport properties of electrons in a metal in the presence of an external magnetic field. The purpose is to examine the application of the variational principle to the solution of the Boltzmann transport equation.

It is shown first that the variational principle gives an automatic representation of the symmetry properties of the thermodynamic functions, even when its numerical validity is poor. This permits the introduction of specific models for the calculation of the momentum distribution function of the electrons through special choices of the approximating functions. Trial functions are, chosen which are linear in the velocity components of the electrons, with coefficients which are expanded in terms of the spherical harmonics adapted to the lattice symmetry. Functions of this type are expected to be reasonable approximations when the free-electron model is nearly adequate. It is shown that this model yields a fairly good representation of experimental work on single crystals, and is also comparable with earlier theoretical work. The detailed calculations refer only to cubic crystals without temperature gradients. *E. L. Hill* (Minneapolis, Minn.)

FLUID MECHANICS, ACOUSTICS

See also 5595, 5649.

5608:

Ter-Krikorov, A. M. Exact solution of the problem of the motion of a vortex under the surface of a liquid. Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 177–200. (Russian)

5609:

Gerber, Robert. Sur une classe de solutions des équations du mouvement avec surface libre d'un liquide pesant. Ann. Inst. Fourier, Grenoble **7** (1957), 359–382.

This paper gives the details, with proofs, of results announced earlier [C. R. Acad. Sci. Paris **242** (1956), 1260–1262; MR **17**, 1247] concerning certain two dimensional, steady, irrotational flows of an incompressible fluid under the action of gravity. The profile S of the bottom of the channel is assumed to be periodic and symmetric; the flows considered are such that the slope of the free boundary profile is opposite to that of S. Let γ denote the acceleration of gravity, ψ_0 the total flux, V_0 the maximum speed, L_0' the half period of the curve S, and Δ the absolute value of the change in the ordinate of S over half a period. Put $H_0=\psi_0/V_0$. The author proves that there exists at least one solution to this type of flow problem, provided that (1) the dimensionless constant $\gamma H_0/V_0^2 \geqq a\pi^2/2$, (2) $H_0/L_0' \leqq c$, and (3) Δ/H_0 is sufficiently small. Here a is a number in the vicinity of 1.2, while c is in the vicinity of 0.5. The proof involves the determination of enough a priori bounds to make possible the application of the Schauder-Leray theory of non-linear functional equations. *D. H. Hyers* (Los Angeles, Calif.)

5610:

Veltkamp, G. W. The drag on a vibrating aerofoil in incompressible flow. I, II. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 278–297.

The flow of an incompressible, inviscid fluid around a thin, flat, vibrating aerofoil of infinite span is determined by using Muskhelishvili's method [Singular integral equations, Noordhoff, Groningen, 1953; MR **15**, 434]. This is effected by expressing the velocity potential in the form

$$\varphi(x, y, t)=\mathrm{Re}[\varphi_1(x, y)e^{ikv}],$$
$$\varphi_1(x, y)=\tfrac{1}{2}[\Omega(z)-\Omega(z^*)],$$

where $\Omega(z)$ is a sectionally holomorphic function of $z=x+iy$. The classical formulae for lift, moment and drag are derived from the behavior of $\Omega(z)$ at infinity. The limiting case of steady motion is also briefly considered.

G. Temple (Oxford)

5611:

Keldysh, V. V. Application of slender body theory to the calculation of aerodynamic properties of low aspect ratio wings with nacelles at their tips. J. Appl. Math. Mech. **22** (1958), 172–181 (126–132 Prikl. Mat. Meh.).

The approximation of Munk and Jones, sometimes called "slender-airplane theory" is applied here to the case of a pointed wing with pointed nacelles of circular section attached symmetrically to its tips. This theory involves solution of plane potential-flow problems in transverse planes. In the present application a sequence of conformal mappings is used to transform the airplane cross section into a figure consisting of three line segments lying on the axis of reals. The potential involves elliptic integrals. Numerical results yielding lift and pitching moment are tabulated and plotted. Comparison is made with the lift of isolated elements to determine the effects of interference.

W. R. Sears (Ithaca, N.Y.)

5612:

Yih, Chia-Shun. Two solutions for inviscid rotational flow with corner eddies. J. Fluid Mech. **5** (1959), 36–40.

The paper studies an idealized simulation of flow in a two dimensional and a symmetrically uniform channel having a sudden contraction. Boundary velocity distributions representing normally viscous flows are assumed, but idealization is made in the flow equations to inviscid flow with a simple sink at the center of a wall across the channel. The closed form solutions available in this setting disclose an inner and outer flow region separated by a boundary surface starting at the wall across the flow channel and asymptotically approaching the outer channel wall upstream. The outer flow resembles corner eddies, but the author is uncertain of the validity of this flow because it does not use the upstream boundary condition directly. Although there may be some question about the overall simulation validity of real viscous flows, it is not clear to the reviewer why the eddy flow validity should be uncertain under the idealized setting.

M. G. Scherberg (Dayton, Ohio)

5613:

Feindt, Ernst-Günther; und Schlichting, Hermann. Berechnung der reibungslosen Strömung für ein vorgegebenes ebenes Schaufelgitter bei hohen Unterschallgeschwindigkeiten. Z. Angew. Math. Phys. **9b** (1958), 274–284.

The application of the Prandtl-Glauert correction to a two-dimensional cascade of thin blades is presented. As is well known, the stagger angle and solidity of the related blade row in incompressible flow are altered. Comparison of the theory with experimental results of Traupel [Mitt. Inst. Therm. Turbomasch. Zürich, no. 3 (1956)] leads to encouraging agreement.

W. R. Sears (Ithaca, N.Y.)

5614:

Haskind, M. D. Oscillation of a cascade of thin sections in incompressible flow. J. Appl. Math. Mech. **22** (1958), 349–354 (257–260 Prikl. Mat. Meh.).

The forces acting on any blade of an infinite, inclined cascade of oscillating aerofoils in incompressible flow are evaluated. The method of conformal transformations is employed to convert the given problem into one involving a vertical cascade. There are no references to English language papers on the subject.

G. N. Lance (Southampton)

5615:

Tretiakov, M. V. Flow round permeable contours. J. Appl. Math. Mech. **22** (1958), 297–304 (220–225 Prikl. Mat. Meh.).

The incompressible irrotational flow of a fluid round a permeable cylinder is discussed. At the boundary of the cylinder the normal component of velocity is assumed to be continuous, but not the pressure, the discontinuity being proportional to the velocity of penetration of the fluid. Accordingly, the boundary condition on the cylinder is non-linear.

The problem is attacked using a vortex distribution $\gamma(s)$ on the cylinder and is elegantly reduced to the solution first of a nonlinear and then of a linear integral equation. On applying the theory of integral equations described by N. I. Muskhelishvili, [Singular integral equations, transl. by J. R. M. Radok, Noordhoff, Groningen, 1953; MR **15**, 434] the author is finally able to transform the equation into one of Fredholm's type.

As an example he solves the problem of the flow with circulation and uniform velocity V_∞ at infinity past a permeable circular cylinder. A puzzling feature of this solution, however, is the behaviour of γ as $v_\infty\to 0$, whence it appears that the stated formulae for γ in terms of a certain parameter K_0 are not consistent.

K. Stewartson (Durham)

5616:

Rogers, Ruth H. The structure of the jet-stream in a rotating fluid with a horizontal temperature gradient. J. Fluid Mech. **5** (1959), 41–59. (1 plate)

Two concentric cylinders have fluid between them, and the system is rotated about the common vertical axis with a constant angular velocity. The inner cylinder is cooled and the outer is heated. It has been found experimentally by Hide [Philos. Trans. Roy. Soc. London. Ser. A **250** (1958), 441–478] that when a certain non-dimensional parameter Θ is greater than about 0.4, the motion, relative to the rotating apparatus, is symmetrical. The paper under review is an attempt to provide a theoretical approach to the asymmetrical case $\Theta<0.4$. In fact, most of the motion is concentrated into a narrow stream which travels from cylinder to cylinder forming a number of lobes. This stream is in some ways analogous to the jet stream in the atmosphere. The author derives the appropriate equations of motion, which contain the parameter Θ, and solves them by expanding in powers of Θ. A jet-stream appears as a particular solution of the first approximation and this is valid except where the stream approaches either cylinder. Finally, it is shown that the equations of motion of an atmosphere can be written in an identical form so a similar solution exists.

G. N. Lance (Southampton)

5617:

Michael, D. H. The separation of a viscous liquid at a straight edge. Mathematika 5 (1958), 82–84.

In this note, the problem of a slow viscous flow bounded by a rigid plane wall and a free surface, inclined at an angle α with the plane wall, is studied by means of the linearized equation. The admitted solution corresponds to a flow leaving the rigid wall without turning at the edge where the solution is singular.

Y. H. Kuo (Peking)

5618:

Kasahara, Eiji; and Simizu, Masayuki. On the steady motion of a viscous fluid through double pipes. Proc. Fac. Engrg. Keio Univ. 10 (1957), no. 36, 8–16.

The authors consider the "extended" Dirichlet problem for the incompressible, steady, laminar, viscous flow through straight pipes with a multiply-connected cross-section whose boundary curves are (a) confocal ellipses, (b) eccentric circles. After suitable coordinate transformations the solutions are given as formal Fourier series. For case (a) the series terminates after finitely many terms. Some graphical representations of numerical computations are given. No reference to the number of previous works on the problem is made.

W. C. Rheinboldt (Washington, D.C.)

5619:

Honda, M. A theoretical investigation of the interaction between shock waves and boundary layers. J. Aero. Sci. 25 (1958), 667–678.

The boundary layer flow is studied near the point where it first separates from the wall in a shock wave-boundary layer interaction. In the laminar case, a solution is obtained by assuming that the velocity profiles ahead of separation can be expressed as members of a one-parameter family of curves and that downstream of separation they are the same as at separation. Good agreement with experiment is found but no mention is made of the inner layer which plays an important role in other theories of self-induced separation, and the style of the paper makes comparison with them difficult. In the turbulent case the boundary layer is divided into two parts of which the outer is supposed to be controlled solely by inertial forces and the inner satisfies the law of the wall. The agreement with experiment is encouraging.

K. Stewartson (Durham)

5620:

Nickel, Karl. Einige Eigenschaften von Lösungen der Prandtlschen Grenzschicht-Differentialgleichungen. Arch. Rational Mech. Anal. 2 (1958), 1–31.

The author presents a rigorous mathematical derivation of some properties of solutions of the boundary-layer equations for two-dimensional, steady, laminar, incompressible flow. It is assumed that the free-stream motion is irrotational, the wall is fixed, and blowing or suction is normal to the wall. The results depend on certain estimation and uniqueness theorems, which are derived for a general class of parabolic partial differential equations and are of some interest in themselves. The conclusions for the boundary-layer equations are as follows: (i) If the initial velocity profile does not have a maximum inside the boundary layer, then the same is true of all velocity profiles downstream. (ii) If the free-stream velocity $U(x)$ does not decrease, and if the initial profile has a maximum $u_m(x_0)$ such that $u_m(x_0)-U(x_0)=e$, then for any downstream profile the maximum $u_m(x)$ is such that $u_m(x)-U(x)\leq e$. (iii) If $U(x)$ does not increase, and if $u_m(x_0)-U(x_0)=e$, then for any downstream profile the velocity cannot be greater than $U(x_0)+e$. (iv) The shear stress assumes its maximum and minimum values on the initial profile and at the wall. This implies that if the initial profile is monotonic the same is true of downstream profiles, and also that the number of extreme values of the velocity profile cannot increase downstream. (v) If $U'(x)\geq 0$ and if blowing at the wall is excluded, the velocity inside the boundary layer can never become zero. Also, the wall shear stress cannot become zero, so that separation does not occur. If $U'(x)>0$, these conclusions are also true for arbitrary blowing. (vi) For an impenetrable wall, if the gradient $U'(x)$ changes sign G times between x_0 and x, then the final velocity profile has at most $G+1$ more inflection points than the initial profile. $G+1$ can be replaced by G if the initial profile is convex at the wall and $U(x)$ has a relative minimum there, or if the initial profile is concave at the wall and $U(x)$ has a relative maximum there. (vii) For a given free-stream velocity $U(x)$ the entire boundary layer is uniquely determined by the initial velocity profile.

D. W. Dunn (Ottawa, Ont.)

5621:

Napolitano, L. G. The Blasius equation with three-point boundary conditions. Quart. Appl. Math. 16 (1958), 397–408.

The Blasius equation subject to three-point boundary conditions, describing the interaction between two parallel streams, is solved by way of a series in terms of ascending powers of the ratio $\lambda=(u_1-u_2)/u_1$, where the u_i's are the outer streams' velocities.

The first three terms of the series are analytically expressed in terms of the repeated integrals of the complementary error function (i^n erfc η) and of the repeated integrals of the square of the successive integrals of the complementary error function ($j^m i^n$ erfc η). (From the authors' summary.)

H. C. Levey (Nedlands)

5622:

Jain, P. C. Gravitational instability of an infinite homogeneous and stationary turbulent medium. Proc. Nat. Inst. Sci. India. Part A 24 (1958), 230–233.

L'Auteur considère un fluide compressible en mouvement sous l'action d'un potentiel de gravitation. D'après la théorie de Jeans, certains petits mouvements de ce fluide ont la forme d'ondes planes, les fluctuations de densité étant données par

$$\delta\rho=A(t)e^{ik\cdot\gamma}.$$

Ils sont stables si $k^2<4\pi G\bar{\rho}/c^2$ (γ constant adiabatique, c vitesse du son). En astronomie, on est conduit à s'intéresser à des fluides turbulents. En négligeant la viscosité et en utilisant l'hypothèse de quasi-normalité pour les corrélations spatio-temporelles de densité, on retrouve la condition $k^2<4\pi G\bar{\rho}/(c^2+\frac{1}{3}\overline{u^2})$, déjà obtenue par Chandrasekhar, et qui diffère de la condition de Jeans par la présence du terme $\frac{1}{3}\overline{u^2}$.

J. Bass (Paris)

5623:

Coburn, N. The method of characteristics for a perfect compressible fluid in general relativity and non-steady Newtonian mechanics. J. Math. Mech. 7 (1958), 449–481.

The aim of the investigation is two-fold: firstly, the theory of characteristics for a gas-flow in general relativity is to be developed; secondly, by degeneration of the resulting equations to Newtonian mechanics, the theory of characteristics is to be extended to non-steady classical flows. In the first part of the paper a quasi-linear system

of necessary conditions is found in terms of directional derivatives along bicharacteristic and related directions which are consequences of the flow equations. These are obtained for the case when the square of the velocity of light is three times the square of the speed of sound. The conditions are expressed by equations of three types; first, a set of equations defining the bicharacteristic and related directions; second, the quasi-linear flow equations in terms of characteristic variables; third, certain integrability conditions. The next step is to show that a class of coordinate systems exists in space-time defined by the vectors associated with the flow. The degeneration to Newtonian mechanics is made in two steps: one step takes the theory from general to special relativity and is thus equivalent to neglecting gravitational effects; the second proceeds from special relativity to Newtonian theory and thus gets rid of the effects of high velocities. The result is a set of twenty-four equations which constitute the Newtonian characteristic system for a non-steady rotational isentropic flow of a compressible fluid. An existence theorem is verified for a two-dimensional irrotational, isentropic flow of a polytropic gas.

G. C. McVittie (Urbana, Ill.)

5624:

Mackie, A. G. A direct method of using the hodograph plane in fluid dynamics. Proc. Edinburgh Math. Soc. 11 (1958/59), 107–114.

An introductory exposition of known results concerning discontinuous motions. *T. M. Cherry* (Melbourne)

5625:

Ryzov, O. S. Some degenerate transonic flows. J. Appl. Math. Mech. 22 (1958), 355–361 (260–264 Prikl. Mat. Meh.).

Il s'agit essentiellement de l'étude d'écoulements transsoniques présentant un caractère d'homogénéité (solutions semblables); ces solutions dépendent en définitive de la résolution d'une équation différentielle non linéaire du premier ordre, qui est formée ici dans le cas d'écoulements tridimensionnels. Les principales applications sont envisagées pour des écoulements plans ou de révolution, mais aucune application nouvelle n'est développée. Par un procédé de passage à la limite, en faisant croitre indéfiniment l'ordre d'homogénéité, de nouvelles solutions particulières sont obtenues. *P. Germain* (Paris)

5626:

Fraenkel, L. E. A two-dimensional air intake in a sonic stream. J. Fluid Mech. 4 (1958), 629–649.

Cette étude est développée dans le cadre de la théorie des petites perturbations appliquée aux écoulements transsoniques. L'obstacle est formé de deux demi plans s'étendant à l'infini aval et schématisant ainsi une entrée d'air. A l'amont l'écoulement est sonique; à l'aval, à l'intérieur de la région limitée par les deux plans, la vitesse est connue et supposée légèrement subsonique. L'écoulement dans la région du plan s'étendant jusqu'à la frontière transsonique est déterminé par la résolution d'un problème aux limites posé dans le plan de l'hodographe dont la solution est explicitement formulée. L'écoulement à l'infini et au voisinage de l'arète est étudié avec l'aide des solutions homogènes transsoniques. Ces résultats sont appliqués à la détermination de la trainée du dispositif et au calcul effectif des distributions de pression le long de l'axe et le long des parois. *P. Germain* (Paris)

5627:

Garabedian, Paul R. Applicazione al flusso supersonico del problema di Cauchy per un'equazione ellittica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 24 (1958), 282–286.

The author has previously described [J. Math. Phys. **36** (1957), 192–205; MR **19**, 706] a scheme for numerical construction of a plane compressible flow with a prescribed detached shock wave. Roughly speaking, it depends on interpreting one of the independent variables as a complex variable and thereby replacing the elliptic equation for the stream function by a hyperbolic equation. In this paper the basic ideas for this procedure are developed for the simpler equation $\Delta\psi=\psi^2$ with initial data on $x=0$ or on $x^2-y^2=1$. Discussion of the latter initial curve suggests that from the form of the initial curve and initial data one can determine qualitatively the location and nature of the singularities of ψ.

J. H. Giese (Aberdeen, Md.)

5628:

Shmyglevskii, Iu. D. Supersonic profiles with minimum drag. J. Appl. Math. Mech. 22 (1958), 368–374 (269–273 Prikl. Mat. Meh.).

The shape of two-dimensional profiles having a minimum drag in a supersonic flow is computed. In any general case, the solution of the variational problem must be performed numerically and it is found, for all the examples considered, that to all intents and purposes the profile is wedge shaped. *G. N. Lance* (Southampton)

5629:

Legendre, Robert. Écoulement supersonique linéarisé autour d'un cône de faible section. C. R. Acad. Sci. Paris **247** (1958), 2094–2096.

L'auteur étudie les écoulements supersoniques linéarisés autour d'une aile conique sans s'autoriser les simplifications usuelles pour l'écriture de la condition de glissement le long de l'obstacle. Ce dernier, en particulier, n'est pas "aplati" sur sa surface moyenne; ceci évidemment fait disparaître les singularités de bout d'ailes, inhérents à la solution classique, mais complique sérieusement l'expression de la solution. L'auteur propose un mode nouveau de représentation du potentiel des vitesses adapté aux exigences du problème ainsi formulé.

P. Germain (Paris)

5630:

Busemann, Adolf. Aus- und Eintrittsstösse an Schaufelgittern. Z. Angew. Math. Phys. **9b** (1958), 191–202.

Steady compressible flow relative to a rotating turbine or compressor wheel is considered in the regime where the resultant speed relative to a blade is supersonic but the axial speed is subsonic, so that conditions are pseudo-subsonic as in the case of a swept wing. Continuity and momentum conservation lead to relations between conditions before and behind the blade row. These are interpreted geometrically. For perfect-gas flow a construction is given which leads to curves of states attainable through a given cascade. The different states have different entropies (as well as energies) and, by entropy considerations, certain parts of the curves must be replaced by discontinuities for entrance and exit processes, respectively. Some of the entrance processes indicated by the diagrams of the paper seem especially complicated. The discontinuities may be combinations of Hugoniot shocks (shock waves) and Carnot shocks. The application of these methods to stages is discussed.

W. R. Sears (Ithaca, N.Y.)

5631:

Miles, John W. On panel flutter in the presence of a boundary layer. J. Aero./Space Sci. **26** (1959), 81–93, 107.

In previous papers [J. Aero. Sci. **23** (1956), 771–780; **24** (1957), 107–118; MR **18**, 253, 839] the author considered the flutter of plane and cylindrical panels in a supersonic air stream. The results then obtained predicted a greater degree of instability than has been observed; one reason may be the presence of a boundary layer which was not taken into account in the theory. The present paper investigates this possibility by considering the flutter of an infinite panel in the presence of a shear flow. It is shown that typical boundary layers may reduce the negative damping by an order of magnitude; however, this conclusion is restricted to long monocoque cylinders for which the critical wavelength and the boundary layer thickness are of the same order of magnitude. *G. N. Lance* (Southampton)

5632:

Kulikovskiĭ, A. G. On Riemannian waves in magnetic hydrodynamics. Dokl. Akad. Nauk SSSR **121** (1958), 987–990. (Russian)

L'auteur étudie les solutions de la magnéto-aérodynamique pour lesquelles les fonctions inconnues dépendent uniquement des valeurs d'une certaine fonction $\varphi(x, t)$; il obtient ainsi deux familles de solutions. Les variations des inconnues en fonction de l'une d'elles (la masse spécifique) se déterminent par quadratures. L'évolution du fluide au cours du temps est ramenée à l'étude du signe de la dérivée de la vitesse de propagation des ondes par rapport à la masse spécifique. Pour l'une des familles de solutions les ondes ne se déforment pas, pour l'autre famille elles donnent naissance soit à des ondes simples soit à un choc. *H. Cabannes* (Marseille)

5633:

Golitsyn, G. S. Plane problems in magnetohydrodynamics. Soviet Physics. JETP **3(47)** (1958), 473–477 (688–693 of Russian original).

The author considers the plane flow of an inviscid compressible perfectly conducting fluid in a magnetic field which is directed normal to the flow plane. Conditions for potential motion, $\nabla\times\mathbf{v}=0$, are deduced. The magnitude of the magnetic field is proportional to the density so that the analysis reduces to the usual hydrodynamical problems with an effective sound velocity $c_m^2=d(P+\frac{1}{2}h^2)/d\rho$ replacing the ordinary sound velocity c.

In particular, the Prandtl-Meyer flow for a conducting gas is studied in detail. It is found that the maximum possible turning angle of the vector velocity in the rarefaction wave decreases for increasing field strength. *H. Greenspan* (Cambridge, Mass.)

5634:

Ludford, G. S. S. On initial conditions in hydromagnetics. Proc. Cambridge Philos. Soc. **55** (1959), 141–143.

The relaxation time for initial conditions in a good fluid conductor to approach those of a perfect conductor is obtained. The result is essentially the same as that for conventional solid conductors. *A. A. Blank* (New York, N.Y.)

5635:

Bazer, J. Resolution of an initial shear-flow discontinuity in one-dimensional hydromagnetic flow. Astr. J. **128** (1958), 686–712.

En magnéto-aérodynamique les discontinuités de contact ne comportent pas de discontinuités de vitesse, chaque fois que la composante du champ magnétique normale à la surface de discontinuité n'est pas nulle. Cette remarque amène l'auteur à etudier le mouvement d'un fluide compressible doué d'une conductivité électrique infinie, en contact avec une plaque plane animée subitement d'un mouvement de glissement sur elle même. La solution du problème est obtenue par combinaison d'un choc exceptionnel (Switch-On Shocks) et d'un écoulement par ondes simples concourantes lentes; ces questions sont étudiées au préalable. *H. Cabannes* (Marseille)

5636:

Carstoiu, John. Sur les équations fondamentales de la magnéto-hydrodynamique et quelques-unes de leurs applications. C. R. Acad. Sci. Paris **247** (1958), 1716–1718.

A partir des équations classiques de le magnéto-hydrodynamique, l'auteur établit qu'une particule chargée douée d'un tourbillon initial peut créer un champ magnétique, de direction fixe et d'intensité proportionnelle au temps; il pense pouvoir donner ainsi une explication de la formation de la terre. *H. Cabannes* (Marseille)

5637:

Kakutani, Tsunehiko. Effect of transverse magnetic field on the flow due to an oscillating flat plate. J. Phys. Soc. Japan **13** (1958), 1504–1509.

L'auteur étudie l'écoulement d'un fluide visqueux incompressible doué de conductivité électrique, autour d'une plaque plane infinie animée d'un mouvement vibratoire, en présence d'un champ magnétique constant perpendiculaire à la plaque. Des formules générales sont établies et quatre cas limites (correspondant aux faibles valeurs et aux grandes valeurs de la pression magnétique et du nombre de Reynolds magnétique) sont étudiés. *H. Cabannes* (Marseille)

5638:

Segre, S. On the formation of magneto-hydrodynamic shock waves. Nuovo Cimento (10) **9** (1958), 1054–1057.

L'auteur considère les équations de la magnéto-aérodynamique dans le cas des mouvements rectilignes non stationnaires; la conductivité électrique étant supposée infinie, il construit une solution très simple qui représente la propagation d'une onde de choc plane. *H. Cabannes* (Marseille)

5639:

Kemp, Nelson H.; and Petschek, Harry E. Two-dimensional incompressible magneto-hydrodynamic flow across an elliptical solenoid. J. Fluid Mech. **4** (1958), 553–584.

This paper concerns the two dimensional steady flow of an incompressible constant conductivity fluid through an elliptically shaped solenoid containing a uniform magnetic field directed perpendicular to the flow plane. The effects of the Hall current and ion slip are included in the generalized Ohms law,

$$\mathbf{j}=\sigma[\mathbf{E}+\mathbf{q}\times\mathbf{B}-\kappa\mathbf{j}\times\mathbf{B}+\lambda(\mathbf{j}\times\mathbf{B})\times\mathbf{B}];$$

where κ is the Hall coefficient and λ the ion slip coefficient. The analysis is based on a double perturbation expansion in the magnetic Reynolds number and the ratio, S, of the magnetic force per unit area to the dynamic pressure. Solutions including first order terms in each parameter are obtained for all quantities of interest. Several graphs illustrate the variation of the force and moment on the solenoid, with angle of attack, fineness ratio, h, of the ellipse, Hall current and ion slip.

Some of the interesting results of this analysis are as follows. (i) The zero order current inside the ellipse is constant. (ii) The vorticity is confined to the interior of the ellipse and to the wake, which is bounded by the zero order streamlines just tangent to the ellipse. (iii) To first order in S, there is a moment and a force; the latter is antiparallel to the zero order force. (iv) To first order in the magnetic Reynolds number there is a moment but no force. (v) For κ and λ both zero, the lift is a maximum for a fineness ratio of about 0.3. (vi) As κ increases ($\lambda=\kappa^2/500$, $h=1$) the lift first increases and then decreases. Both the drag and the total force decrease monotonically.

H. Greenspan (Cambridge, Mass.)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

5640:

Focke, Joachim. Zur wellenoptischen Abbildung in Systemen mit grosser relativer Öffnung. Opt. Acta **4** (1957), 124–126.

The new expressions for the energy density and the energy flow in an electromagnetic field, introduced in the scalar theory of H. S. Green and E. Wolf [Proc. Phys. Soc. Sect. A **66** (1953), 1129–1137] are used to determine the structure of the aberration-free diffraction image formed in unpolarized light in systems of various angular apertures. It is also shown that, in contrast with the classical scalar theory of optics, the new scalar theory is fully consistent with energy conservation.

E. Wolf (Manchester)

5641:

Polovin, R. V.; and Tsintsadze, N. L. Circular waves in an electron-ion beam. Soviet Physics. JETP **34(7)** (1958), 440–443 (637–642 of Russian original).

The authors calculate the circular oscillations of an electron-ion beam in a perfectly conducting cylindrical waveguide of circular cross-section. They assume that the magnetic field produced by the beam is so strong that the cyclotron frequencies of the electrons and ions are small compared with the frequency of oscillations. From the Maxwell field equations and the equations of motion and continuity they obtain upon linearization a differential equation whose eigenvalues and eigenfunctions yield respectively the frequencies and amplitudes of the circular oscillations. By qualitative consideration of this equation they show that the beam is stable with respect to circular oscillations and find the frequency limits within which such oscillations may occur.

C. H. Papas (Pasadena, Calif.)

5642:

di Francia, G. Toraldo. Babinet's principle for diffraction at a plane screen with directional conductivity. Nuove Cimento (10) **9** (1958), 309–315.

The conventional form of Babinet's principle for electromagnetic waves considers the similarity between a) the diffraction around a perfectly conducting infinitesimally thin plane disk of arbitrary shape, and b) the diffraction through a hole in a perfectly conducting screen, the hole having the same contour as the former disk. This paper deals with the analogous similarity between c) the diffraction around a plane anisotropic disk in each point of which the conductivity is infinite in the direction along a special trajectory passing through this point, and zero in the direction normal to it, and d) the diffraction through the same plane if its part outside the anisotropically conducting disk has been replaced by an isotropic perfectly conducting screen. The problem is treated in terms of a coordinate z perpendicular to the plane, and two curvilinear coordinates connected with the continuous set of the above mentioned trajectories in this plane. The relations between the electric and magnetic fields and a Hertzian vector potential for problem c) are identical, apart from differences with respect to signs, with the corresponding relations between the magnetic and electric fields and an associated magnetic (or Fitzgerald) vector potential for problem d). The similarity between c) and d) also holds for the differential equation for the distribution of these vector potentials across the anisotropic disk. The electric and magnetic fields in case c) thus prove to be interchangeable, apart from constant factors, with the magnetic and electric fields in case d).

H. Bremmer (Eindhoven)

5643:

Zin, Giovanni. Elettromagnetismo maxwelliano e principio di Huyghens. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 421–427.

Formulae are given relating the electromagnetic field within a bounded region D to that in an unbounded region with a surface distribution of charges and currents on the boundary of D. *V. M. Papadopoulos* (Providence, R.I.)

5644:

Malyužinec, G. D. The excitation, reflection and radiation of surface waves in a wedge-like region with given face impedances. Dokl. Akad. Nauk SSSR **121** (1958), 436–439. (Russian)

A study is made of the wave field $p(r, \varphi)$ in a wedge-shaped region $r>0$, $-\Phi<\varphi<\Phi$ with boundary condition $\partial p/r\partial\varphi \pm ikz_0p/z_\pm=0$ on the faces $\varphi=\pm\Phi$ of the wedge. The ratios $z_0/z_\pm$ of the wave impedance in the medium to the normal surface impedance of the boundaries $\varphi=\pm\Phi$ are the sines of the Brewster angles $\theta_\pm$. For absorbing boundaries, $0<\operatorname{Re}\theta_\pm\leq\frac{1}{2}\pi$. The exact field is obtained by Sommerfeld's theory, and an asymptotic expression for large r by the saddle-point method. The physical significance of the different terms in the solution and of various limiting cases is discussed.

R. N. Goss (San Diego, Calif.)

5645:

Karal, Frank C., Jr.; and Karp, Samuel N. Diffraction of a skew plane electromagnetic wave by an absorbing right-angled wedge. Comm. Pure Appl. Math. **11** (1958), 495–533.

The paper presents an exact solution of the electromagnetic problem of obtaining the field diffracted by a three-dimensional right-angled wedge when a plane wave is incident from an arbitrary angle. The special feature of the problem is that one surface of the wedge is absorbing; that is, an impedance boundary condition is prescribed on that surface and the authors show that this condition means complete absorption for one direction of the incident wave and partial absorption for other angles of incidence. The second surface of the wedge is perfectly conducting.

The problem cannot be solved by the method of separation of variables. Instead the authors adapt a method due to J. J. Stoker [Quart. Appl. Math. **5** (1947), 1–54; MR **9**, 163] and H. Lewy [Bull. Amer. Math. Soc. **52** (1946), 737–775; MR **9**, 163] and previously used for water wave problems. The essence of the method is to introduce

two new functions,

$$\mathscr{E}_y=-\lambda E_y+\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y},\ \mathscr{E}_z=-\lambda E_z-\frac{\partial E_x}{\partial z}+\frac{\partial E_z}{\partial x},$$

wherein E_x, E_y, and E_z are the components of the desired electric field, $\lambda=i\omega\mu/z$, and z is the impedance in the impedance boundary condition. It is shown that the functions $\mathscr{E}_y$ and $\mathscr{E}_z$ satisfy simple boundary conditions on the two surfaces of the wedge and each satisfies the wave equation. These functions can be obtained by the method of separation of variables and from them the desired functions E_x, E_y, and E_z can be obtained by direct integrations. The $\mathscr{E}_y$ and $\mathscr{E}_z$ selected as solutions of their respective problems must possess singularities at the edge of the wedge but of the lowest order possible in order that E_x, E_y, and E_z possess the singularity generally accepted to be the correct one for wedges in which both surfaces are perfectly conducting.

In addition to the exact solution an asymptotic representation of the far field is given. *M. Kline* (Aachen)

5646:

Beck, G.; and Nussenzveig, H. M. Uncertainty relation and diffraction by a slit. Nuovo Cimento (10) **9** (1958), 1068–1076.

Heisenberg's uncertainty relation is discussed for the example of the diffraction of a vertically incident monochromatic plane wave $\sin(kz)\exp(-i\omega t)$ through a slit that is situated in the plane $z=0$ in the x direction. This wave may represent an incident particle, or also the amplitude of a linearly polarized incident wave. The spatial frequency distribution in planes z=constant is determined by the Fourier transform with respect to y, $\chi(k_y, z)$ say, of the solution $\varphi(y, z)$ for the diffracted field. The uncertainties of the coordinate y and of the frequency (wave number) k_y could be defined as root mean square values Δy and Δk_y derived with the aid of probability densities proportional to $|\varphi(y, z)|^2$ and $|\chi(k_y, z)|^2$ respectively. The consideration of limiting cases then shows that either Δy or Δk_y may become infinite. However, these quantities always prove to be finite if defined as the half-widths (for constant z) of the functions $|\varphi(y, z)|^2$ and $|\chi(k_y, z)|^2$. The case of a wide slit leads to $\Delta k_y(z)\sim 1/a$ (a=width of slit); this is in accordance with the special value $\Delta k_y(\infty)\sim 1/a$ derived from the usual interpretation of the uncertainty relation for the diffraction observed at infinity. *H. Bremmer* (Eindhoven)

5647:

Kear, George. The scattering of waves by a large sphere for impedance boundary conditions. Ann. Physics **6** (1959), 102–113.

The treatment of the diffraction by a spherical object is considerably simplified by introducing a boundary condition of the form $\partial u/\partial r=kZu$ at the surface $r=a$ of the sphere (Leontovich boundary condition or impedance boundary condition). The problem in question then only concerns the space outside the sphere. In the case of a point source the total field can be expanded in Legendre functions $P_n(\cos\vartheta)$ of integral orders n. This expansion can be reduced, with the aid of a "Watson transformation", [see, e.g., A. Sommerfeld, "Partielle Differentialgleichungen der Physik", Akademischer Verlag, Leipzig, 1947; MR **10**, 195; p. 284; H. Bremmer, "Terrestrial radio waves", Elsevier, New York-Amsterdam-London-Brussels, 1949; MR **11**, 295; p. 31] into a new series of Legendre functions (the mode series or residue series) the orders $\nu_1, \nu_2, \cdots$ of which constitute a discrete spectrum of eigenvalues depending on the above boundary condition. However, this Watson transformation can also be applied to the scattered field only, that is, to the difference of the total field and the primary field (existing in the absence of the sphere). The scattered field then is proved to consist of the sum of a residue series and an integral over ν. The latter can be split into an integral from $\nu=0$ to a special eigenvalue $\nu=\nu_L$, and another integral from ν_L along a path connecting all eigenvalues of orders exceeding L. An application of Euler-Maclaurin's summation formula shows that this latter integral contribution is asymptotically cancelled by the higher-order terms of the residue series (provided that L is not too small). The main field contribution is thus proved to arise from the integral from $\nu=0$ to $\nu=\nu_L$, and from the first L terms of the residue series. This leads to convenient analytical representations when $ka\gg 1$. The calculations of the article are specialized to the case of both source and point of observation at infinity, in opposite directions; this corresponds to forward scattering due to a plane incident wave. In particular, explicit numerical formulas (derived by taking $L=3$) are represented for the limiting cases $Z=0$ and $Z=\infty$. *H. Bremmer* (Eindhoven)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 5562, 5604.

5648:

Farzetdinov, M. M. On the uniqueness of the solutions of the equation of weak convection in the steady state. J. Appl. Math. Mech. 22 (1958), 393–397 (286–288 Prikl. Mat. Meh.).

The following system of equations is considered:

$$(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla p-\nabla\times\nabla\times\mathbf{v}-\lambda_\gamma\theta,$$
$$\sigma\mathbf{v}\cdot\nabla\theta=\Delta\theta,\quad \operatorname{div}\mathbf{v}=0.$$

Here, $\mathbf{v}$ denotes the velocity of the fluid, θ the difference between the temperature of the fluid and its mean value θ^*. γ is the unit vector in direction of the gravitation. λ and σ denote the Grashof and Prandtl numbers, respectively. The fluid occupies a bounded volume D, with boundary S in an infinite rigid solid medium with temperature θ^0 (also measured from θ^*). The boundary conditions are: $\mathbf{v}=0$, $\theta=\theta^0$, $d\theta/dn=\alpha(d\theta^0/dn)$ ($\alpha>0$) on S; and $\nabla\theta^0\to\mathbf{a}$ for $\rho\to\infty$ ($\mathbf{a}$ unit vector). — Under the assumption that the solutions $\mathbf{v}$, p, θ, θ^0 can be expanded in series of powers of the Grashof number λ, the author proves the uniqueness of these expansions by investigating the successive linear equations which are obtained by substituting the series into the equations of the problem (p is determined up to a constant).

J. C. C. Nitsche (Minneapolis, Minn.)

5649:

Burgers, J. M.; and Ghaffari, A. On the application of steam-driven water jets for propulsion purposes. J. Res. Nat. Bur. Standards **60** (1958), 137–141.

The authors consider the momentum that can be added to a water jet by condensing into it a jet of high-velocity steam through a constant pressure, no-shock process. A typical calculation is made and it is shown that if this system is used to drive an underwater vehicle so constructed that the water jet is obtained from the surrounding water, then the efficiency of the system, in terms of

power needed for the production of the steam, is of the order of $\frac{1}{2}$%. Consideration is also given to the amount of water jet dispersion necessary to provide, for typical conditions, sufficient surface area for conduction into the water interior of the heat of condensation of the condensed steam. It is found that it would be necessary to provide a dispersion into drop sizes of less than 2 mm diameter.
P. Chiarulli (Chicago, Ill.)

QUANTUM MECHANICS

See also 5576, 5606, 5607, 5706.

5650:

Keller, Joseph B. Corrected Bohr-Sommerfeld quantum conditions for nonseparable systems. Ann. Physics **4** (1958), 180–188.

Approximate semi-classical solutions of the Schrödinger equation for non-separable systems are constructed in such a way as to give the correct form of the "quantum conditions" with the appropriate integer, half integer or other quantum number. Besides giving a quantization rule in agreement with quantum mechanics the approximate solution of the Schrödinger equation — which can be constructed by classical mechanics — may prove to be useful.
M. Cini (Rome)

5651:

Baranger, Michel. Simplified quantum-mechanical theory of pressure broadening. Phys. Rev. (2) **111** (1958), 481–493.

A general formulation of the theory of collision (pressure) broadening of spectral lines is proposed. The aim is to treat the radiating system and its collisions with neighboring molecules by quantum mechanical methods. The starting point is the assumption that in both the initial and final states of the radiating molecule there exist potential functions which determine the interactions with neighboring molecules. Collisions between the excited molecule and surrounding molecules are then treated by the methods of quantum mechanical scattering theory. The major part of the discussion is concerned with the detailed analysis of the approximations to be made and their validity. Only non-degenerate energy levels of the radiating molecule are considered in the present paper.
E. L. Hill (Minneapolis, Minn.)

5652:

Baranger, Michel. Problem of overlapping lines in the theory of pressure broadening. Phys. Rev. (2) **111** (1958), 494–504.

The theory of pressure broadening of spectral lines is treated for the case of overlapping lines. The discussion follows the lines of the author's earlier work [preceding review] but the considerations are simplified by treating the perturbing molecules as prescribed systems which follow the trajectories they would have according to Newtonian mechanics. The radiating molecule is considered to have a Hamiltonian operator which is time-dependent. With certain approximations this can be replaced by a time-independent non-hermitian Hamiltonian operator. The broadenings and shifts of the spectral lines are determined by the (complex) eigenvalues of this latter operator. In the case of overlapping lines the broadened lines may be asymmetrical in shape. The theory is given only in a general form and is not reduced to actual applications.
E. L. Hill (Minneapolis, Minn.)

5653:

Ritus, V. I. Invariant representations of the scattering matrix. Soviet Physics. JETP **6** (1958), 972–974.

The scattering matrix S for the reaction $a+b\to a'+b'$ is expressed by invariant angular operators and finally expressed in terms of a finite number of spin operators Q_i, each invariant under rotations and reflections. The scattering matrix can be written in the form

$$S(k', k)=\sum A_i(k', k)Q_i(k', k, T).$$

This form is convenient to use in studying the general properties of scattering matrices, investigating angular distributions of reactions, and similar problems. A method for constructing the Q_i is given, and their number is determined for a reaction with given initial and final spins. The restrictions placed upon the form and the number of the Q_i by the condition that the scattering matrix be invariant under time reversal are considered.

For inelastic processes, this does not restrict the matrix S. Several examples, meson nucleon scattering, nucleon nucleon scattering, photomeson productions, and compton scattering are given in which the S is represented by the Q_i's.
M. Cini (Rome)

5654:

Regge, T. On the analytic behaviour of the eigenvalue of the S-matrix in the complex plane of the energy. Nuovo Cimento (10) **9** (1958), 295–302.

A study on the analytic behavior of the eigenvalues of the S-matrix in the complex plane of the energy $E=k^2$ is here carried out for the case of spinless nonrelativistic particles scattered by a fixed center of force. A method is developed capable of finding in a finite number of steps any false zero or singularity of the eigenvalues of the S-matrix. The same method shows that in the upper k-plane Jost's $f(k)$ function may have not only simple poles but also any kind of singularity. Our method enables us to establish the main features of these singular points. (From the author's summary)
M. Cini (Rome)

5655:

Källén, Gunnar. The concept of particles in quantized field theories. Proc. Math. Phys. Soc. Egypt **5** (1956), no. 4, 101–111 (1957). (Arabic summary)

The two versions of the concept in quantum field theory of states with a definite number of particles are reviewed and their relationship discussed. The older idea, that associated with eigenstates of the Hamiltonian for uncoupled fields, was long the sole basis for constructing the solution of problems with interaction by perturbative or Tamm-Dancoff methods. The more recent concept of incoming (outgoing) particle states, suggested by physical consideration of collision problems, is directly a characterization of the situation when interactions are present. Its significance is illustrated with the help of two well-known exactly soluble models, the neutral scalar field in interaction with a fixed source and the Lee model.
A. Klein (Philadelphia, Pa.)

5656:

Lomont, J. S. Dirac-like wave equations for particles of zero rest mass, and their quantization. Phys. Rev. (2) **111** (1958), 1710–1716.

The author deals with an algebraic (nonspinor) theory, which takes the photon as a model for all massless particles. The wave equations of his theory have the Dirac

form

$$\alpha_\mu \nabla^\mu \psi = 0, \quad \mu = 0, 1, 2, 3,$$

where the α's are square Hermitian matrices satisfying

$$[\alpha_i, \alpha_j]_+ = 2\delta_{ij}, \quad i, j = 1, 2, 3.$$

For every positive half-integral spin there is an equation. The dimension of the matrices of the spin s equation is only $4s$. The equations are gauge-independent, only transverse particles (states) enter the theory. The spin $\frac{1}{2}$ and spin 1 equations are the 2-component neutrino and photon equations, respectively. Massless particles with more than one spin state are also described by these equations. The single-spin equations are quantized. The infinite-dimensional equations are not considered.
M. Pinl (Cologne)

5657:

Blohinčev, D. I. On a possible limit of applicability of quantum electrodynamics. Nuovo Cimento (10) 9 (1958), 925–929.

It is shown that certain four-fermion processes (weak interactions) lead to cross-sections which increase faster with energy than electromagnetic processes with the same initial states. This leads to the proposition that quantum electrodynamics breaks down at energies where these cross-sections become comparable ($\sim$2 Mev in the center of mass system for electron-electron or electron-photon scattering).
F. Rohrlich (Baltimore, Md.)

5658:

van Kampen, N. G. Causality and the Kramers-Kronig relations. II. Nederl. Tijdschr. Natuurk. 24 (1958), 29–42. (Dutch)

This is a continuation of an earlier review paper by the same author [same Tijdschr. 24 (1958), 1–14; MR 19, 1129], giving a non-technical description of the theory and application of causality arguments and dispersion relations. The previous discussion, which was entirely classical, is first applied to the scattering of classical fields and then extended to include also the non-relativistic Schrödinger equation. In this case, there is an essential complication involved as it is impossible to build up a wave packet with a sharp front in the Schrödinger theory of a particle with a finite mass. The causality requirement is therefore reformulated to read: The probability of finding a particle in the outgoing scattered wave is never bigger than the probability that the ingoing wave has hit the scattering center. In particular, this probability is always smaller than or equal to one. This formulation, originally given by the author himself some years ago [Phys. Rev. (2) 91 (1953), 1267–1276; MR 15, 588], allows one to derive analyticity properties of the S-matrix analogous to the results obtained in the theory of the scattering of light. The last section of the paper contains a very short description of the application of similar ideas to quantized field theories. The causality requirement is here given the conventional form that the commutator between two observable field quantities vanishes for space-like separations. Presumably because of the great mathematical complexity in the further development of the theory, no details are given of the formal apparatus of dispersion theory in this field, but some of the most important numerical applications are mentioned.
G. Källén (Lund)

5659:

Johnson, Kenneth A. Consistency of quantum electrodynamics. Phys. Rev. (2) 112 (1958), 1367–1370.

Källén has given an inconsistency proof of quantum electrodynamics by showing without use of perturbation theory that at least one of the renormalization constants is infinite. The present author asks whether these conclusions are consistent with perturbation theory. It is noted that the usual computation of the renormalization constants (using ultraviolet and infrared cut-offs) in perturbation theory with Lorentz gauge is not consistent with Källen's result (when expanded in perturbation theory). However, the author observes that the electron field renormalization constant is not gauge-invariant, so that there might exist a gauge which yields agreement with Källen's result. Such a gauge is shown to exist, provided the electron field renormalization is suitably redefined. The question whether this proposed renormalization when used in Källen's argument would lead to different conclusions is left open, but the author's work emphasizes the important point that Källen's inconsistency proof does not establish the divergence of any gauge-invariant quantity in the theory.
F. Rohrlich (Baltimore, Md.)

5660:

Weidlich, W. Zum Aufbau der Quantenfeldtheorie ohne Lagrangeformalismus. Nuovo Cimento (10) 9 (1958), 228–248.

"It is investigated to what extent the quantum theory of elementary particles can be built up by using the following principle only: "There exists an irreducible representation of the algebra of creation and annihilation operators a^+, a, and the infinitesimal transformations of the inhomogeneous Lorentz group can be constructed out of them." The representation of the inhomogeneous Lorentz group can then be split up in irreducible ones classified by Wigner and describing different sorts of elementary particles. Space-time dependence of all operators is introduced by a unitary transformation of the a^+, a. Energy-momentum tensors and charge-current vectors are found without use of fields. Fields and field-equations, corresponding to the various sorts of elementary particles, may also be set up but are dispensable. Interactions are introduced as unitary transformations of the a^+, a, mixing the irreducible representations of the Lorentz group. This corresponds to in general non-local interaction terms in the Lagrange formalism." (Author's summary)
P. W. Higgs (London)

5661:

Kibble, T. W. B. The commutation relations obtained from Schwinger's action principle. Proc. Roy. Soc. London. Ser. A 249 (1959), 441–444.

Extending earlier work by Kibble and Polkinghorne, Kibble shows here how to construct an action density which is consistent with the Bose-Einstein and Fermi-Dirac statistics of quantised field theory, but not with more general types of statistics.
H. S. Green (Adelaide)

5662:

Klein, Abraham; and McCormick, Bruce H. Derivation of the two nucleon potential. Progr. Theoret. Phys. 20 (1958), 876–889.

An unambiguous method is used for the computation of the internucleon potential to fourth-order terms, in the fixed source gradient coupling theory. The result, which depends on five arbitrary parameters, is stated without details of the intermediate calculation. There is a large P-wave contribution to the central force, and apparently some difficulty in reconciling the theory with experiment.
H. S. Green (Adelaide)

5663:

Hori, S.; and Wakasa, A. Individual mass reversal and parity non-conservation. Nuovo Cimento (10) **6** (1957), 304–310.

The authors suggest that the theory of elementary particles be invariant under a family of transformations which they call individual mass reversals. The transformation on a spinor field ψ_a is: $\psi_a \to \eta_a{}^a \gamma_5 \psi_a$, $\psi_b \to \eta^a{}_b \psi_b \cdots$; $m_a \to -m_a$, $m_b \to m_b$, $\cdots$; i.e., all fields suffer phase changes but ψ_a is multiplied by γ_5 and the sign of the corresponding mass is reversed. Such transformations were also considered by J. Tiomno [Nuovo Cimento (10) **1** (1955), 226–232; MR **16**, 1184], as applying to groups of spinor fields. Tiomno also considered transformations in which all spinor fields are multiplied by γ_5 and all masses reversed in sign. The point of individual mass reversal is to require the separate invariance. Related invariance postulates have been introduced later by R. Marshak and E. Sudarshan [Padua-Venice Conference; Letter to the editor, Phys. Rev. (2) **109** (1958), 1860–1862] and R. Feynman and M. Gell-Mann [ibid. **109** (1958), 193–198; MR **19**, 813]. The authors deduce the forms of interaction of Yukawa and Fermi type consistent with their invariance postulate. For the Fermi interactions the result turns out to be equivalent to the "two component" neutrino theory [T. D. Lee and C. N. Yang, ibid. **105** (1957), 1671–1675; MR **21**#1174; A. Salam, Nuovo Cimento (10) **5** (1957), 299–301 MR **21**#1173; L. D. Landau, Nuclear Phys. **3** (1957), 127–131 MR **21**#1172], provided that all the fields involved are distinct. When two of the fields are the same (as in μ decay) the possible interactions are somewhat more general than the two component theory. The extra generality arises from the arbitrariness of the phases η.

A. S. Wightman (Princeton, N.J.)

5664:

Leite Lopes, J. Note on the interaction of elementary particles. An. Acad. Brasil. Ci. **29** (1957), 521–530.

Fixed source meson theory is used to show that the capture of negative muons does not proceed appreciably through pions, but that a direct four-fermion interaction must be invoked. Applying such a universal Fermi interaction to baryons of different strangeness leads to a prediction of $\Lambda^\circ\beta$ decay that is probably in contradiction with experiment. For the principal mode of K meson decay, a reasonable rate is obtained, assuming the K meson is pseudoscalar coupled to baryons relatively weakly.

S. Bludman (Berkeley, Calif.)

5665:

Cutkosky, R. E. Meson exchange effects in two-nucleon states. Phys. Rev. (2) **112** (1958), 1027–1038.

This paper deals with a new theoretical approach to the two-nucleon problem. It aims at encompassing all phenomena which contain meson exchange effects, and at dealing only with physical particles characterized by experimental quantities. The mathematical formalism used here to achieve these goals is based on the Heitler-London method [Z. Physik **44** (1927), 455–472], which was originally devised to discuss the hydrogen molecule. It is claimed that the method is even more suited for nuclear problems. The actual formalism is derived for fixed source meson theory and the representation is discussed in general terms. The new states become exact eigenstates when the two nucleons are far apart. Matrix elements between these states can be expressed in terms of the properties of isolated nucleons by means of expansions in the number of mesons exchanged. Isobar effects can also be incorporated easily. The application of the method to covariant theories is only sketched, although the claim is made that there should not be any insurmountable difficulties in making this extension. The two appendices give an illustration of the new representation for neutral scalar mesons and discuss the N-meson state of two nucleons.

M. J. Moravcsik (Livermore, Calif.)

5666:

Arnowitt, R.; and Feldman, G. Behavior of the meson-nucleon cross section at high energies. Phys. Rev. (2) **108** (1957), 144–147.

In order to obtain a unique set of dispersion relations for a process like pion-nucleon scattering, one has to make an assumption, among other things, about the behavior of the cross section at high energies. The usual assumption made is that the pion-nucleon cross section approaches a constant, and the meager indirect experimental evidence does not disagree with this assumption. Symanzik [Nuovo Cimento (10) **5** (1957), 659–665; MR **18**, 972] has shown that the cross section must increase less rapidly than the meson laboratory energy ω. The present paper gives a limit on the other side and states that the cross section must increase at least as rapidly as ω^{-1}. The proof assumes that local field theory holds and that the coupling constant λ of pion-pion interaction is not zero. The theorem continues to hold even if certain types of K-meson and hyperon terms are included.

M. J. Moravcsik (Livermore, Calif.)

5667:

Banerjee, Haridas. Scattering of a longitudinally polarised electron beam by a uniform magnetic field. Proc. Nat. Inst. Sci. India. Part A **24** (1958), 279–287.

Polarization effects in the scattering of a longitudinally polarized electron beam by a uniform magnetic field are considered. Two cases are studied, i.e. when the magnetic field is parallel and when it is perpendicular to the incoming beam. In both cases the degree of polarization remains unaltered by the scattering, but the relative orientation between the directions of polarization and momentum changes. For the parallel case the polarization direction does not depend on the magnetic field. For the perpendicular case, however, it does and it also depends on the magnetic moment of the electron. The experimental study of such scattering, therefore, can, at least in principle, lead to a determination of the magnetic moment of the free electron. The calculational technique used in this paper is the standard Feynman-Dyson S matrix formalism.

M. J. Moravcsik (Livermore, Calif.)

5668:

Green, H. S.; and Biswas, S. N. Covariant solutions of the Bethe-Salpeter equation. Progr. Theoret. Phys. **18** (1957), 121–138.

This paper represents one of the few efforts made to obtain rigorous solutions to a Bethe-Salpeter equation for two fermions. The example of charge-independent pseudoscalar meson theory in the ladder approximation is chosen. Though there is good evidence from the study of approximate solutions that this lowest (covariant) approximation is not adequate from a physical point of view, it is nonetheless of great interest and importance to study the mathematical properties of the resultant equation. The most important result obtained is the existence of a discrete spectrum characterized by four constants of the motion, one of which has no non-

relativistic analogue. This confirms a result of Wick and Cutkowsky [G. C. Wick, Phys. Rev. **96** (1954), 1124–1134; MR **16**, 655; R. E. Cutkosky, Phys. Rev. **96** (1954), 1135–1141; MR **16**, 656] obtained for an analogous model of less physical interest. The physical significance of the additional solutions is not understood, though the authors conjecture a possible connection with the "strangeness" quantum number. Other aspects of the investigation include a detailed study of the instantaneous approximation to the interaction, a proof that the particular model gives rise to a repulsive interaction at short distances, and a general method of generating solutions of higher angular momentum from spherically symmetric ones by Lorentz transformation. *A. Klein* (Philadelphia, Pa.)

5669:

Tsuneto, Toshihiko; and Fujiwara, Izuru. Relativistic wave equations with maximum spin two. Progr. Theoret. Phys. **20** (1958), 439–456.

This paper is concerned with the reduction of the first-order spinor equation for hypothetical particles of spin 0, 1 or 2 to a number of tensor equations. It corrects an earlier paper [Prog. Theoret. Phys. **14** (1955), 267–282; MR **18**, 95] and, while it repeats a good deal of work by other authors, it contains a complete and useful statement of results. *H. S. Green* (Adelaide)

5670a:

Bogoliubov, N. N. A new method in the theory of superconductivity. I. Soviet Physics. JETP 34(7) (1958), 41–46. (58–65 Z. Eksper. Teoret. Fiz.)

5670b:

Tolmachev, V. V.; and Tiablikov, S. V. A new method in the theory of superconductivity. II. Soviet Physics. JETP 34(7) (1958), 46–50. (66–72 Z. Eksper. Teoret. Fiz.)

5670c:

Bogoliubov, N. N. A new method in the theory of superconductivity. III. Soviet Physics. JETP 34(7) (1958), 51–55. (73–79 Z. Eksper. Teoret. Fiz.)

Substantial progress has been achieved recently in the development of a theory of superconductivity in metals by the application of quantum mechanical perturbation methods. The problem requires an accurate evaluation of the interactions of the electrons near the top of the Fermi distribution with the lattice vibrations and with each other. The greatest difficulties are those of isolation and calculation of the dominant interaction terms.

Major work in the field has made use of the Hamiltonian operators proposed by H. Fröhlich [Proc. Roy. Soc. London. Ser. A. **215** (1952), 291–298] and by J. Bardeen, L. Cooper, and J. R. Schrieffer [Phys. Rev. (2) **108** (1957), 1175–1204; MR **20** #2196]. The energy operators used by these authors differ to some extent in the manner in which they account for the interactions between the electrons. The three papers under review are concerned with the application of a new scheme of perturbation calculation developed by N. N. Bogoliubov to the Hamiltonian operators given by Fröhlich and by Bardeen et al.

In the first part the new method of calculation is described by Bogoliubov and is applied to the Hamiltonian operator given by Fröhlich. Virtual processes which lead to singular terms in the perturbation calculation are termed "dangerous". In the present problem dangerous (Feynman) diagrams are contributed by the virtual creation of an electron pair near the top of the Fermi distribution without accompanying phonons. Such dangerous terms are grouped together by canonical transformations. A "principle of compensation of dangerous diagrams" is then invoked to permit setting their total contribution to zero, the resulting equations being used to determine the required canonical transformations. When this procedure is applied to Fröhlich's Hamiltonian operator it is found that it leads to a superconducting state and, to the approximation considered, yields substantially the same results as were found by Bardeen et al. {The connection between the perturbation methods used by Bogoliubov and by Bardeen et al is discussed also by K. Yosida [Phys. Rev. (2) **111** (1958), 1255–1256; MR **20** #4386].}

In part II the method given in part I is applied in greater detail to the evaluation of the major contributions of electrons near the top of the Fermi distribution. It is found that to the approximation considered the Hamiltonian operators proposed by Fröhlich and by Bardeen et al lead to equivalent results for the superconducting state.

In part III the Hamiltonian operator given by Bardeen et al is analyzed by a variant of the method of canonical transformations given in part I, called "the method of approximate second quantization". This consists in replacing the original Hamiltonian operator by a new one which contains only those terms which are found to be of greatest importance. The simpler form of the new Hamiltonian permits the use of improved approximation procedures. It is suggested that the new method of calculation will make it easier to evaluate higher approximations and to find the thermodynamic functions.

{Since the discussions given in these papers are still within the ambit of perturbation theory, they do not appear to contribute decisively to questions which have been raised concerning the basic validity of the theory of superconductivity. It is possible, however, that further extension of the method of canonical transformations may be helpful in clarification of the theory.}

E. L. Hill (Minneapolis, Minn.)

RELATIVITY

See also 5623.

5671:

Darwin, Charles. The gravity field of a particle. Proc. Roy. Soc. London. Ser. A **249** (1959), 180–194.

Orbits of particles and of light rays in the Schwarzschild solution of Einstein's gravitational equations are considered. Elegant expressions for the equations of the orbits in terms of elliptic functions are obtained. Chief attention is concentrated on the mathematical problem of determining the nature of the orbits that pass close to the central body, i.e., to within a small multiple of the length m, which is the mass of the central body converted to length-units. This body is regarded as a point-mass at $r=0$ and so no physical significance is attached to the results. The main ones are: No hyperbolic orbit can have perihelion inside $r=3m$ and no elliptic one inside $r=4m$. Particles with such orbits are captured. Circular orbits are possible and are unstable if $3m<r<4m$, in the sense that one kind of disturbance leads to capture, the opposite to ejection in a spiral to infinity. Circular orbits

for which $4m<r<6m$ are unstable, disturbances either leading to capture or to a spiral path with an aphelion distance greater than the radius of the original circle and with final return to it. Circular orbits with $r>6m$ are stable. No light-ray from infinity can escape capture unless its initial asymptotic distance is greater than $3\cdot 3^{\frac{1}{2}}m$. If a field of stars could be viewed beyond the point-mass central body, the stars would appear to be accompanied by faint "ghosts" on both sides of the central body. These ghosts would all lie just outside the distance $3\cdot 3^{\frac{1}{2}}m$.

G. C. McVittie (Urbana, Ill.)

5672:

Kronsbein, John. Relativity in a stationary spherical or elliptic space. Phys. Rev. (2) **112** (1958), 1384–1391.

In a recent publication [Phys. Rev. (2) **109** (1958), 1815–1822; MR **19**, 1237], the author developed the fundamentals of the theory of relativity in static spherical and elliptic space from a new point of view. The present paper is a continuation of (and must be read together with) this memoir, and deals with the theory of relativity in a stationary spherical or elliptic space which differs significantly from Einstein's space but appears to be the natural and simplest generalisation of the latter. Again, coordinates distinct from those used by Einstein are introduced, and the concepts of non-euclidean geometry form the basis of the author's methods. It is found that the stationary space under consideration is no longer entirely spherically symmetrical in all directions, and there are two different types distinguished by right- and left-handed Clifford translations of the reference systems. The differential equations of the geodesics are evaluated: these are distorted as compared with the Einstein space, an effect which an observer may interpret as the result of gravitational forces which resemble those produced by the accelerated "Einstein elevator" in Minkowski space (but in spherical or elliptic space constant Clifford velocity will generate the field). The Lorentz group links all such spaces; hence agreement can be reached between observers in any two spaces with regard to the motion of a particle observed by both.

H. Rund (Durban)

5673:

Nariai, Hidekazu; and Ueno, Yoshio. On the tests of gravitational theories in terms of an artificial satellite. Progr. Theoret. Phys. **20** (1958), 703–714.

The authors show that Singer's proposal [S. F. Singer, Phys. Rev. (2) **104** (1956), 11–14; MR **18**, 782] to test the general theory of relativity by means of a gravitational red-shift experiment employing artificial satellites will not contribute to a decision between Einstein's general theory of relativity and alternative theories of gravitation such as Whitehead's and Birkhoff's theories. This is because Singer's predicted effect consists of relativistic Doppler effect and gravitational red-shift; the former is common to all theories that are at least Lorentz-covariant, the latter will occur in any theory in which the frequency of a photon will be affected by its gain or loss of potential gravitational energy. The authors also show that any effect based on so-called geodesic deviation is unlikely to serve the purpose of discriminating between the theories enumerated above.

{The reviewer would like to add that in his opinion the desirability of performing such experiments as the one suggested by Singer does not depend merely on their suitability for deciding between Einstein's, Whitehead's, and Birkhoff's theories.}

P. G. Bergmann (New York, N.Y.)

5674:

Miyatake, Yoshio. Perturbational calculations of propagators of the elementary particles interacting with gravitational field. Progr. Theoret. Phys. **20** (1958), 476–486.

The propagation of a field is calculated in perturbation theory for the case of interaction with the (linearized) gravitational field. A cutoff is introduced and the diagrams consisting of iterated lowest self-energy diagrams summed. This procedure leads to "corrected" propagators (whose one virtue is to vanish rapidly for high momenta in the case of derivative coupling) which are then used to discuss the divergences of the s-matrix.

S. Deser (Waltham, Mass.)

5675:

Papapetrou, A. Über periodische Gravitations- und elektromagnetische Felder in der allgemeinen Relativitätstheorie. Ann. Physik (7) **1** (1958), 186–197.

In a previous publication [Ann. Physik (6) **20** (1957), 399–411; MR **19**, 1020] the author considered non-singular solutions $\mathfrak{g}^{\mu\nu}$ of the field equations $R_{\mu\nu}=0$, where these solutions satisfy the following requirements: (a) $\mathfrak{g}^{\mu\nu}$ is a periodic function of the time; (b) the metric tends to that of the flat Minkowski world as $r\rightarrow\infty$, where r is the distance of a point from a certain "central region" of the field, so that for sufficiently large distances $r\geqq r_0$ the field is weak; while (c) only such fields were considered which are weak throughout the space. De Donder's condition $\mathfrak{g}^{\mu\nu}{}_{,\nu}=0$ was also stipulated, and it was shown that such solutions are trivial in the sense that they may be obtained by means of suitable coordinate transformations applied to the metric tensor of the Minkowski world. In the present paper this result is generalised: (1) condition (c) is dropped, so that fields are considered which may be strong for $r<r_0$; (2) a combined gravitational and electromagnetic field is considered, whose field equations result from the Lagrangian $\mathfrak{g}^{\mu\nu}R_{\mu\nu}+\frac{1}{2}\kappa\mathfrak{g}^{\mu\alpha}\mathfrak{g}^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}$. It is assumed that the gravitational field still satisfies conditions (a) and (b), while the electromagnetic field ϕ_α is to have the same period as $\mathfrak{g}^{\mu\nu}$. It is shown for both cases (1) and (2) that such fields must be time-independent in the region $r\geqq r_0$ of the weak gravitational field, in the sense that the time-dependence of $\mathfrak{g}^{\mu\nu}$ and that of ϕ_α may be removed by means of coordinate and gauge transformations respectively.

H. Rund (Durban)

5676:

Goldberg, J. N. Conservation laws in general relativity. Phys. Rev. (2) **111** (1958), 315–320.

In a fresh discussion of conservation laws resulting from the general invariance of the Lagrangian, the author derives two hierarchies of divergence-free affine tensors ("pseudo-tensors") in general relativity theory. One hierarchy includes the canonical pseudo-tensor, the other the Landau-Lifshitz affine tensor [L. Landau and E. Lifshitz, "The classical theory of fields", Addison-Wesley Press, Cambridge, Mass., 1951; MR **13**, 289; chapter 11]. The different tensors in each hierarchy are distinguished by their weights under affine transformation. Physical significance and the related problem of angular momentum conservation are both discussed.

F. A. E. Pirani (Chapel Hill, N.C.)

5677:

Bergmann, Peter G. Conservation laws in general relativity as the generators of coordinate transformations. Phys. Rev. (2) **112** (1958), 287–289.

In a (classical) field theory, an infinitesimal transfor-

mation $y_A \to y_A + \delta y_A$ of the field variables y_A, if it leaves the form of the (first-order) Lagrangian L invariant, may be generated by a generating density C^ρ, to which L is related by (1) $L^A \delta y_A + C^\rho{}_{,\rho} = 0$ where $L^A \equiv \partial L/\partial y_A - (\partial L/\partial y_{A,\rho})_{,\rho}$ and $f_{,\rho}$ means $\partial f/\partial x^\rho$ [see P. G. Bergmann and R. Schiller, Phys. Rev. (2) **89** (1953), 4–16; MR **14**, 606].

Here such transformations are induced, in Einstein's theory of gravitation with field equations (2) $G^{\mu\nu}=0$, by coordinate transformations, which clearly leave the Lagrangian form-invariant. Eq. (1) shows that the generating density (which is undetermined up to the divergence of an arbitrary skew field $V^{[\rho\sigma]}{}_{,\sigma}$) is divergence-free whenever the field equations (2) are satisfied.

The author exploits these results to construct a wide class of conserved affine tensors which depend on an arbitrary vector field and its first partial derivatives and on the metric tensor and its first partial derivatives. His construction substantially extends some results of J. N. Goldberg [#5676 above] and contributes notably to the understanding of the concept of conservation laws in general relativity theory.

F. A. E. Pirani (Chapel Hill, N.C.)

5678:

Iyengar, V. Non-singular static solutions in Bonnor's modified generalised theory of gravitation. Rev. Fac. Sci. Univ. Istanbul Sér. A 22 (1957), 25–29. (Turkish summary)

The author proves that the field-equations of a generalized theory of gravitation attributed to W. B. Bonnor (no reference to Bonnor's work is given) do not admit any non-singular static solutions representing a non-vanishing total mass. *G. C. McVittie* (Urbana, Ill.)

5679:

Takeno, Hyôitirô. A comparison of plane wave solutions in general relativity with those in non-symmetric theory. Progr. Theoret. Phys. 20 (1958), 267–276.

The author obtains plane wave solutions for the field equations of general relativity and the non-symmetric form of unified field theory. The solutions obtained have the property that in the latter theory the electromagnetic field does not affect the metric of space-time. This result is one of the disturbing predictions of the last form of unified field theory proposed by Einstein.

M. Wyman (Edmonton, Alta.)

5680:

Hlavatý, V. Basic principles of the unified field theory of the second kind. I, II. J. Math. Mech. 7 (1958), 323–354, 833–866.

In Einstein's unified field theory the equations

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ik}{}^\alpha g_{\alpha j} + \Gamma_{kj}{}^\alpha g_{i\alpha} \tag{1}$$

are used to define the basic linear connection $\Gamma_{jk}{}^i$. In several previous papers Professor Hlavaty has discussed the existence and uniqueness of the solutions of (1).

We write (2) $h_{ij} = \frac{1}{2}(g_{ij}+g_{ji})$, (3) $k_{ij} = \frac{1}{2}(g_{ij}-g_{ji})$, (4) $g_{ij} = h_{ij}+k_{ij}$, (5) $g=|g_{ij}|$, and (6) $h=|h_{ij}|$. In this notation the solution of (1) is unique if $g(g-2h)\neq 0$ and $h<0$. The author has previously exhibited the explicit solution of (1) under these conditions. Professor Hlavaty has now written two papers investigating the degenerate case $h=0$, $g\neq 0$. The work of the two papers has been divided into five chapters. The first paper, containing two chapters, develops the tensorial algebra and analysis required to attack the problem. The second paper, containing the remaining three chapters, is devoted to obtaining the explicit expression for the components of the linear connection $\Gamma_{jk}{}^i$.

In papers of this length involving a multitude of notation it is not possible to adequately describe the methods used or the results obtained. It suffices to say that these papers exhibit the same attention to both detail and clarity that are characteristic of the many papers of the author on the same subject.

M. Wyman (Edmonton, Alta.)

5681:

Rzewuski, J. On a possible geometrical interpretation of gauge transformations. Nuovo Cimento (10) 9 (1958), 942–949.

[The page numbers refer to the book, V. Hlavaty, "Geometry of Einstein's unified field theory", Noordhoff, Groningen, 1958; MR **20** #5067]. The local vector space of the space-time is mapped in a three-dimensional projective spinor space on a linear complex $\mathscr{K}_3$ (p. 241). Each individual vector is mapped on two lines of $\mathscr{K}_3$ which coincide (and constitute a congruence $\mathscr{K}_2 \subset \mathscr{K}_3$) for null vectors. (p. 230). The ensuing spinor coordinate transformation are gauge transformations G (p. 237). The above mentioned pair of mapping lines has four coordinates in common.

The author considers four-dimensional spinor space and admits also spinor coordinate transformations T which change the vector coordinates. He chooses for the above mentioned four mapping coordinates a bilinear function of z^1z^2 and $\bar{z}^3$, $\bar{z}^4$ with two arbitrary parameters μ, ν. He obtains in this way a mapping which is a generalization of the mapping of lineal elements (p. 257–258). The transformations T preserve a scalar product only for the special relation $\mu = \pm\nu$. The final remark of the paper concerns (without proofs) the irreducible representation of an enlarged group. *V. Hlavatý* (Bloomington, Ind.)

ASTRONOMY

See also 5622, 5635, 5671.

5682:

Batten, Alan H. The effect of reflection on the determination of masses of close binary systems. Monthly Not. Roy. Astr. Soc. 117 (1957), 521–533.

The author has considered a particular effect caused by "reflection" in close binary systems. Part of the light of each star in such a system is intercepted by the other, but it must all be re-radiated into space, if these stars are to maintain their radiative equilibrium. As a result, the hemisphere of each star which faces the other will be brighter than the respective other hemispheres. This will cause the effective light centre of each stellar disk to be displaced in such a way that the velocity observed by the spectroscope will be lower than the true orbital velocity of the centre of mass of each star. In consequence, the mass of the system will be underestimated from radial velocity measurements which have not been corrected for this effect. Furthermore, since the magnitude of the effect on each star is proportional to the relative luminosity of its companion, it follows that the mass ratio of the system will also be incorrectly estimated. It is the author's purpose to investigate the magnitude of the errors involved, and he derives a phase law giving the correction that should be applied to observed orbital velocities at any phase.

In his work he restricts himself to the case of two spherical stars, rotating as rigid bodies (but not necessarily about axes perpendicular to the orbital plane). He also supposes that the "reflection" obeys Lambert's Law. He gives estimates for the actual magnitude of the correction in a number of special cases, and shows that it is at least as sensitive to the relative luminosities of the two stars as it is to their relative separation. In very close systems (i.e. separation not more than about three times the radii of the component stars) the correction may be of the same order as the observed mass itself. The author's results cannot be reliably extrapolated this far, however, since he takes no account of the distortion, and consequent gravity darkening, which is bound to be present in such close systems. *Z. Kopal* (Manchester)

5683:

King-Hele, D. G. The effect of the earth's oblateness on the orbit of a near satellite. Proc. Roy. Soc. London. Ser. A. **247** (1958), 49–72.

The perturbations of a near satellite are evaluated in spherical coordinates for arbitrary inclinations and to the fourth order in the eccentricity in near-circular orbits ($e<0.05$). Beginning with an elliptic orbit at an arbitrary inclination about a spherical earth, a solution to the second order in e is obtained in the equations of motion with acceleration components $\partial U/\partial r$, $\partial U/r\partial\theta$, $\partial U/r \sin\theta\partial\phi$ (U being the potential of a non-uniform oblate spheroid in which the gravitational field is independent of longitude and symmetrical about the equatorial plane) by assuming a uniform motion of the node. The solution yields the amount of motion of the node and the equation of an ellipse with perturbed true longitude.

A solution to the third order is then found by seeking a term in the motion of the node factored by e. Unfortunately, the choice of variables is such that the motion of the node comes out undefined for values of the longitude which are integral multiples of π. The author suggests that such discontinuities arise "because the orbit of the satellite momentarily passes out of the orbital plane as defined here, and could probably be avoided by appropriate re-definition". Assuming the motion of the perigee to be $O(e^2)$, the author finds, in this approximation, that it is constant and has the factor $(5\cos^2\alpha-1)$, where α is the inclination, now a well-known result, showing that apsidal motion vanishes for an inclination of $\tan^{-1} 2=63.4°$.

Finally a fourth-order solution is found by seeking terms to the second order of the eccentricity in the motion of the node and terms to the fourth order in the radius vector. The motion of perigee is identical to the value obtained in the previous approximation.

Next the author examines the results of this theory. The secular perturbations to the second order are studied in terms of the synodic period and the harmonic mean value of the radius vector

$$\frac{1}{\bar{r}}=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{r}\,d\psi$$

where ψ is the longitude in the orbital plane measured from the point of highest latitude (that is, 90° minus the argument of latitude). The secular motions of the node and perigee are

$$\frac{d\Omega}{dt}=J\sqrt{\frac{g}{R}}\left(\frac{R}{\bar{r}}\right)^2\left(\frac{R}{a}\right)^{1.5}\cos\alpha+O(J^2),$$

$$\frac{d\beta}{dt}=J\sqrt{\frac{g}{R}}\left(\frac{R}{\bar{r}}\right)^2\left(\frac{R}{a}\right)^{1.5}\frac{5\cos^2\alpha-1}{2}+O(Je).$$

From these the main features of the motion may be deduced. The perturbation in the radius vector to the second order produces a mean distance 13 nautical miles larger in a polar orbit than in an equatorial one.

A direct solution is then obtained for orbits in the equatorial plane in terms of elliptic integrals. The results agree with the previous perturbation theory when $\alpha=0$. The author then discusses earlier papers on the effects of oblateness and finally gives a comparison of his theory with Sputniks 1 and 2. The agreement is good except for a slight discrepancy in $d\Omega/dt$ for Sputnik 2 which is attributed to drag. *M. S. Davis* (New Haven, Conn.)

5684:

Merman, G. A. Recherches qualitatives dans le problème des trois corps. Byull. Inst. Teoret. Astr. **6** (1958), 687–712. (Russian. French summary)

On étudie dans ce mémoire les mouvements finales dans le problème des trois corps pour le cas des valeurs négatives de la constante des forces vives. Les conditions suffisantes de la décomposition du mouvement des trois corps en deux mouvements presque indépendants et approchés de ceux de Kepler sont obtenues; on trouve aussi les limites de la distance et de la vitesse du corps éloigné de deux autres. *Résumé de l'auteur*

5685:

Merman, G. A. Sur la représentation de la solution générale du problème des trois corps par des séries convergentes. Byull. Inst. Teoret. Astr. **6** (1958), 713–732. (Russian. French summary)

On montre la possibilité de la représentation de la solution générale du problème des trois corps par les séries convergentes et distinctes de celles de Sundman à l'aide de quelques estimations déduites des théorèmes générales de l'existence et du prolongement des solution des équations différentielles en utilisant le travail rapporté ci-dessus. *Résumé de l'auteur*

5686:

Popović, Božidar. Special perturbations of the vectorial elements of planetoid orbits. Bull. Soc. Math. Phys. Serbie **8** (1956), 47–52. (Esperanto. Serbo-Croatian summary)

The author eliminates the velocity from the Herrick-Popović equation for the eccentricity-vector in favor of the radius-vector and the derivative of the area-velocity. He arrives at a combination of equations which, if properly coded, can successfully be used in the computations of special perturbations, leading to results more directly than Musen's method.

P. Musen (Cincinnati, Ohio)

5687:

Simon, R. L. Sur la stabilité vibrationnelle des oscillations non radiales d'une étoile gazeuse. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 610–621.

For the sake of simplicity, the author limits himself to the discussion of the stability of a non-radial oscillation in a gaseous star in radiative equilibrium. The radiation pressure is supposed to be negligible compared with the gas pressure. He assumes that the standing vibrations of such a star are linear and adiabatic. He then gives the expression for the coefficient of vibrational stability for non-radial oscillations which, under the above assumptions, has the same form as for radial oscillations. The star is stable or unstable with respect to a particular oscillation according as this coefficient is positive or

negative. From this expression the author shows that the degeneracy which affects the adiabatic proper value problem remains complete, even when the generation of nuclear energy and radiative conductibility are taken into account. The author regards this as his most important conclusion.

He then turns to consider a particular non-radial oscillation of the standard model. This is the oscillation termed "f" by Cowling. He shows that the standard model is much more stable with respect to this oscillation, than with respect to any of the radial oscillations considered by Ledoux. He supposes that this remains true for any model. At any rate the influence of energy generation, so long as it is confined to the central regions of the star, is even weaker for non-radial oscillations than for radial oscillations. Moreover, contrary to the radial case, stability with respect to non-radial oscillations increases as the exponent of the temperature in the energy generation law increases. *A. H. Batten* (Manchester)

GEOPHYSICS

See also 5616.

5688:

Sarkisyan, A. S. On the theory of unsteady wind flows on a homogeneous ocean. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 1232–1237. (Russian)

The work reported here represents a generalization of the results of a previous paper by the same author [Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 1008–1019; MR **19**, 718] in that variation of the Coriolis parameter (previously assumed constant) with latitude is now permitted and sphericity of the earth is also introduced. The investigation of the solution for the flow velocity components and the ordinate of the free surface of the ocean relative to a quiescent surface level then proceeds, as in the earlier paper, by successive approximations, the effect of the Coriolis variation being accounted for by the use of appropriate expansions in terms of associated Legendre polynomials. *J. F. Heyda* (Cincinnati, Ohio)

OPERATIONS RESEARCH AND ECONOMETRICS

See also 5269, 5546.

5689:

Sargan, J. D. The instability of the Leontief dynamic model. Econometrica **26** (1958), 381–392.

The author studies the Leontief dynamic model in which the demand for current input j by industry i is $a_{ji}x_i(t)$, where a_{ji} is a constant, $x_i(t)$ the output of industry i, and the demand for input j on capital account is $b_{ji}(dx_i/dt)$, so that the total demand for commodity j is

$$(1)\qquad \sum_1^n a_{ji}x_i(t)+\sum_1^n b_{ji}(dx_i/dt)+y_j(t),$$

where $y_j(t)$ is final demand. A moving equilibrium can be defined by assuming that demand (1) equals supply, i.e., $x_j(t)$. It is argued that this system is unstable in the following sense: if a one-parameter family of systems is constructed which becomes the equilibrium system when the parameter is zero but which admits inequalities of supply and demand when the parameter is positive, then the system has, for parameter values close to zero solutions which diverge from that of the equilibrium system. For example, the system,

$$(2)\qquad x_j(t+\epsilon\mu_j)=\sum_{i=1}^n a_{ji}x_i(t)+\sum_{i=1}^n b_{ji}(dx_i/dt)+y_j(t),$$

with the parameter ϵ, has a positive characteristic root which approaches $+\infty$ as ϵ approaches zero.

K. J. Arrow (Stanford, Calif.)

5690:

Kao, Richard C.; and Rowan, Thomas C. A model for personnel recruiting and selection. Management Sci. **5** (1959), 192–203.

A general model is developed "to determine a strategy — the number of people to be recruited and the minimum acceptable score" in a given test having a known fixed correlation between test and the criterion of job success — "which will yield a minimum cost subject to a given probability that at least a fixed number of good employees will be hired". *M. J. Beckmann* (New Haven, Conn.)

5691:

Chow, Tse-Sun. Operational analysis of a traffic-dynamics problem. Operations Res. **6** (1958), 827–834.

Some solutions are obtained for the equations of traffic-dynamics proposed by Chandler, Herman and Montroll [Operations Res. **6** (1958), 165–184; MR **20** #770].

G. Newell (Stockholm)

5692:

Grenander, Ulf. On heterogeneity in non-life insurance. II. Skand. Aktuarietidskr. **1957**, 153–179 (1958).

[For part I, see same Actuarietidskr. **40** (1957), 71–84; MR **19**, 1243.] In this part the author gives some ways to determine the risk distribution: Maximum likelihood estimates; a straightforward approach; least squares estimates; "Optimum" estimates; estimation with side condition. Bias of the estimates; numerical illustrations.

W. Saxer (Zürich)

5693:

Grenander, Ulf. Some remarks on bonus systems in automobile insurance. Skand. Aktuarietidskr. **1957**, 180–197 (1958).

In Schweden werden die Prämien für die Automobilversicherungen auf Grund von Bonus-Systemen festgelegt. Der Verf. zeigt, wie durch mathematische Schätzung des Risikos und Einteilung der Policen in Gruppen ein möglichst gerechtes Bonussystem konstruiert werden kann und untersucht auch die Gewinnmöglichkeiten.

W. Saxer (Zürich)

5694:

Åkerberg, Bengt. On educational annuity, non-dependent and in combination with endowment assurance. Skand. Aktuarietidskr. **1957**, 145–152 (1958).

In order to avoid negative premium reserves for educational annuities, the author deduces a simple criterion with the aid of which it is possible, in the most common cases, to decide the longest acceptable time for the payment of premiums. *W. Saxer* (Zürich)

5695:

Thullen, Peter. Über den relativen Beharrungszustand einer Bevölkerung. Mitt. Verein. Schweiz. Versich.-Math. **58** (1958), 177–196.

Der Verfasser betrachtet relativ-stationäre (konstante Altersstruktur) und stabile Bevölkerungen (konstante Vermehrungsrate) und führt bereits vorhandene Unter-

suchungen über den Beharrungszustand durch Lotka, den Referenten und andere weiter. Er beweist die folgenden Sätze.

Eine Bevölkerung ist dann und nur dann stabil, falls die Eintrittsfunktion eine zeitlich unveränderliche Altersstruktur und zugleich eine konstante Wachstumsrate besitzt, welche gleich der Wachstumsrate der Bevölkerung ist.

Eine Bevölkerung mit der Häufigkeitsfunktion $L(x; t)$ befinde sich im Zeitintervall $[t_0, t_1]$ im relativen Beharrungszustand. Ist dann die Altersstruktur der durch die Funktion $N(\xi, \tau)$ bestimmten Neueintritte in $x_0 \leq \xi \leq u$, $t_0 \leq \tau \leq t_1$ konstant, so ist die gegebene Bevölkerung in $[t_0, t_1]$ stabil.

Ist in einer relativ-stationären Bevölkerung die relative Verteilung nach Verbleibensdauer jeweils für jedes vorgegebene Alter zeitlich konstant, so ist die Bevölkerung stabil. *W. Saxer* (Zürich)

5696:

van Klinken, J. On some simple stochastic processes of special use in actuarial statistics. Verzekerings-Arch. Actuar. Bijv. **35** (1958), 107–117.

The author describes some important processes occurring in actuarial practice. Examples: Widow pensions, invalidity-pensionholders, numerical illustrations.

W. Saxer (Zürich)

5697:

Fürst, Dario. Il caso limite del problema della rovina dei giocatori nell'ipotesi di riserva limitata. Giorn. Ist. Ital. Attuari **20** (1957), 120–143.

A Brownian motion with forward drift and with an absorbing barrier at $x=0$ and a reflecting barrier at $x=1$ is an interesting model for the fluctuating reserve of an insurance company that consumes as profit any reserve in excess of a predetermined critical amount. The paper carefully studies this stochastic process as a boundary and initial value problem for the corresponding Fokker-Planck equation, mainly by the technique of separation of variables. Connection is stressed with the space-discrete analogue of the same problem, which the author has treated previously [same Giorn. **19** (1956), 63–83; MR **19**, 186.] Qualitative changes in behavior with changes in the ratio of the drift and diffusion coefficients are brought out. *L. J. Savage* (Rome)

5698:

Radner, Roy. The application of linear programming to team decision problems. Management Sci. **5** (1959), 143–150.

Suppose spending A dollars results in the production of xA, and spending B dollars generates the demand yB. If product and demand are perishable, and if the cost of obtaining capital beyond a certain limit is sufficiently high, the profit function is convex and polyhedral. Best values of A (and similarly B) are obtained for the cases where x and y are known, where x is known and a probability distribution for y is known, and where a joint distribution is known. Generalizations are indicated, and it is also shown how such a problem can be formulated as a linear programming problem.

T. E. Hull (Pasadena, Calif.)

5699:

Gessford, John. Scheduling the use of water power. Management Sci. **5** (1959), 179–191.

The problem studied is that of a hydroelectric plant which endeavors to meet a preassigned demand for power in such a way as to minimize the amount of power which must be supplied by a supplementary source such as a thermal plant. The demand for power during each week of the planning period is a known quantity, the inflow of water into the reservoir each week is a stochastic variable with a known probability distribution, the cost of procuring supplementary power and the penalty for failing to meet demand are also known data. A simple graph for determining the optimal amount of water to use the first week is given and, with minor modifications, the same graph can be used in later weeks as soon as the amount of water in storage at the beginning of the week is known.

Slight variants of this problem are also considered. For example, if the inflow of water is not stochastic, it is shown that the optimal policy is to generate power hydraulically to the maximum extent permitted by the contents of the reservoir.

Proofs are omitted for the most part, but may be found in J. Gessford and S. Karlin, "Optimal policy for hydroelectric operation," Studies in the mathematical theory of inventory and production, pp. 179–200.

R. Dorfman (Cambridge, Mass.)

5700:

✱Berge, C. Théorie générale des jeux à n personnes. Mémor. Sci. Math., no. 138. Gauthier-Villars, Paris, 1957. 114 pp.

This medium-size treatise gives a set-theoretic, treatment of a collection of fundamental theorems on n-person games. Most of the theorems are existing ones, though there are some additions. In this case, "n-person" is taken to include "two-person" games as well; for instance, the min-max theorem is given. Reformulation, generality, and elegance, rather than discursiveness or completeness, characterize the book. There are, however, a few examples to illustrate the concepts when needed. Though the material is not in historical order, the author is scholarly about first sources, and has listed almost fifty of them in the bibliography.

There are five chapters: I. Games with complete information; II. Topological games; III. Games with incomplete information; IV. Convex simultaneous games; and V. Coalitions. The chapters after the first are almost but not quite independent. A game (of perfect information) is an n-partition of an abstract set X, a mapping (multivalued) of X into itself, and n quasi-ordering relations on X. The game is said to be a "jeu de paiement" if the relations can be represented by n bounded real-valued functions on X. A player i can be active or passive. If S is the subset of positions encountered in the play of the game, the gain of an active player is $\sup_{x \in S} f_i(x)$ and of a passive player is $\inf_{x \in S} f_i(x)$.

In chapter I, it is shown that a preferentially finite game (perfect information) of bounded length possesses an equilibrium point. This is a somewhat generalized form of the theorem of Zermelo-von Neumann. There are some results from the theory of graphs, an extension of the theorem to games which are locally finite, and some conditions for the existence of Grundy functions. Chapter II is devoted partly to the notion of a semi-continuous mapping. The X_i are separable topological spaces, the mapping of X into itself is continuous, and the preference functions are semi-continuous. This chapter contains mostly the original work of the author. The theorems concern the topological properties of the set of positions and of the space of strategies which will guarantee a given gain. Under certain not too restrictive conditions (the

game being locally finite, for one), there exists only one good strategy. Chapter III introduces the concepts of simultaneous game, mixed strategy, information set, equivalent schemes of information, perfect recall, behavior strategy, and signaling strategy. The values achieved are compared. Chapter IV contains the proof of a quite general form of the (von Neumann-Mash) equilibrium point theorem (where quasi-concavity replaces linearity), using the Kakutani fixed point theorem (generalized to locally convex linear spaces). There are also several extensions of the theorem, and a few theorems on equilibria which weaken topological conditions and strengthen the convexity condition. There is then the question of determining the good strategies in a finite simultaneous game. The two-person Shapley-Snow theorem is given, though it is a little more difficult to prove than is indicated. Chapter V is an efficient enumeration of the effects of coalitions. The notion of equilibrium point is generalized. Characteristic functions are introduced, and McKinsey's theorem on their equivalence is proved. There is a nice presentation of the proof of uniqueness and existence of the Shapley value, and lastly, the von Neumann-Morgenstern solution.

Both in language and style, the book will appeal to connoisseurs of Bourbaki.

I. Mann (Santa Monica, Calif.)

INFORMATION AND COMMUNICATION THEORY

See also 5535.

5701:

Solodov, A. V. Statistical analysis of nonstationary processes in linear systems by using inverse simulating devices. Avtomat. i Telemeh. 19 (1958), 312–324. (Russian. English summary)

The author extends the Laning-Battin method of adjoint systems [Random processes in automatic control, McGraw-Hill, New York, 1956; MR 18, 74] to nonstationary systems subjected to non-stationary inputs. Let $W(t, \xi)$ be the impulsive response of a time-varying system. By an inverse system the author means a system whose response at time ξ to a unit impulse applied at time zero is given by $W(t, \xi)$ for a fixed t. Expressions of the form

$$\sigma^2 = \int_0^t \int_0^t W(t, \xi) W(t, u) K(u, \xi) d\xi du$$

are evaluated by constructing an inverse system and simulating $K(u, \xi)$ as the solution of a differential equation.

L. A. Zadeh (New York, N.Y.)

5702:

★Elias, Peter. List decoding for noisy channels. Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., Rep. No. 335, 12 pp. (1957).

The list decoding of the title means that the receiver lists L messages rather than one, where $L<M$ (the total possible messages). Some equivocation is left after decoding since if the correct message is in the list L there is assumed to be no error. This paper discusses rates of transmission and the approach to the idea of Shannon's coding theorem for such a system.

R. W. Hamming (Murray Hill, N.J.)

5703:

Caregradskiĭ, I. P. On the capacity of a stationary channel with finite memory. Teor. Veroyatnost. i Primenen. 3 (1958), 84–96. (Russian. English summary)

A. I. Hinčin [Uspehi Mat. Nauk (N.S.) 11 (1956), no. 1(67), 17–75; MR 17, 1098; translated, by R. A. Silverman and M. D. Friedman, as "Mathematical foundations of information theory", Dover Publications, Inc., New York, 1957; MR 19, 1148] pointed out that in the formulation of some arguments it was essential for their validity to understand the capacity of a channel as its ergodic capacity, i.e., the supremum, over all the matching ergodic sources, of the rate of transmission of information through this channel. The paper under review proves that the ergodic capacity of a non-anticipatory stationary channel is equal to its stationary capacity, obtained when the supremum is taken over all the matching stationary sources instead of all the ergodic sources. This clarifies several delicate points in information theory. The proof is constructive and depends largely on the theory of Markov chains.

S. K. Zaremba (Swansea)

5704:

★Zaregradski, I. P. [Caregradskiĭ, I. P.] Eine Bemerkung über die Durchlasskapazität eines stationären Kanals mit endlichem Gedächtnis. Arbeiten zur Informationstheorie II, pp. 65–77. Mathematische Forschungsberichte. VI. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. 77 pp. DM 14.40.

German translation of article reviewed above.

5705:

Joshi, D. D. A note on upper bounds for minimum distance codes. Information and Control 1 (1958), 289–295.

This paper gives 5 theorems on upper bounds for the minimum distance of both unrestricted and group codes. Most of the theorems are not new but proofs are simpler than those in the generally accessible literature.

R. W. Hamming (Murray Hill, N.J.)

5706:

Berger, L. Quantité d'information, et systèmes physiques. Helv. Phys. Acta 31 (1958), 159–166.

The author summarizes the definitions of the entropies $H(X)$, $H(Y)$ and $H(X, Y)$ for the case where X and Y are random variables with integrable joint density function. He then introduces $H(X)+H(Y)-H(X, Y)$ which he denotes by $I_{x}y$ and which he calls the information concerning Y relative to X. This quantity, which is a measure of information received through a channel in standard information theory, is applied to certain probability distributions in classical and quantum mechanics and conservation equations are derived. (Notation is nonstandard and confusing.)

R. A. Leibler (Wheaton, Md.)

CONTROL SYSTEMS

See also 5701.

5707:

Bedel'baev, A. K. On stability of non-linear self-regulating systems. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. no. 6(10) (1957), 51–59. (Russian. Kazah summary)

The author points out that the applicability of certain

criteria of stability to control systems depends on the condition that the object of control be naturally stable when the servomotor is disengaged. The concept of natural stability of the control object is connected with the structure of a certain quadratic form which permits the determination of the stability of the system when the servomotor is cut out. A method for finding the coefficients of the required quadratic form is given. The use of the suggested method is illustrated in an example.

H. P. Thielman (Ames, Iowa)

5708:

Andreev, N.I. On determination of an optimal dynamic system. Avtomat. i Telemeh. **19** (1958), 1077–1090. (Russian. English summary)

Necessary and sufficient conditions are derived for the extremum of a functional that has been selected as a criterion for the comparison of different dynamic systems. Two examples are given to illustrate the proposed method of selecting optimal dynamic systems.

H. P. Thielman (Ames, Iowa)

5709:

Moisil, Gr. K. Isomorphism of relay-contact schemas. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1(49)** (1957), 87–97. (Russian)

This paper is concerned with (idealized) relay contact schemes of sequential type, operating in discrete time intervals, such that if the positions of all contacts are known at any moment, then the positions of all contacts at the next moment are uniquely determined. If such a device has n contacts, then it is capable of assuming 2^n states; we suppose these numbered in some fashion from 1 to 2^n. If the device is started in some initial state, then the states it assumes thereafter are uniquely determined; and the sequence of their numbers will be periodic (except for possible anomalies at the beginning). The sequence of state(-number)s in such a case is called a process. Two processes are called isomorphic if they differ only by a permutation of the state-numbers; and two schemes are isomorphic if a suitable fixed permutation carries any process of one into an isomorphic process of the other. The paper illustrates these concepts with examples, but gives no theoretical discussion.

H. B. Curry (University Park, Pa.)

HISTORY AND BIOGRAPHY

See also 5119, 5120.

5710:

✶Vogel, Kurt. Vorgriechische Mathematik. I. Vorgeschichte und Ägypten. Mathematische Studienhefte, 1. Hermann Schroedel Verlag KG, Hannover; Verlag Ferdinand Schöningh, Paderborn; 1958. 80 pp. DM 8.00.

The first part of this book, the beginnings in prehistoric times, occupies fourteen pages of the text and discusses geometry, ornamentation, and some ancient number systems. Illustrations of all these are given in considerable profusion.

The remainder of the book is concerned with the mathematics of Egypt. The source material is the same as that used by Neugebauer [Vorgriechische Mathematik, Springer, Berlin, 1934]. The author discusses some history of Egypt, Egyptian writing, the mathematical texts, and number and measure systems. A table exhibits the place of the texts in our chronology and in the Egyptian dynastic system. Computation with whole numbers and fractions is described and illustrated by problems, largely from the Rhind and Moscow papyri. Aside from the "unit fractions," the use of a symbol for $\frac{2}{3}$ is mentioned. Gardiner [Egyptian Grammar, second edition, Oxford Univ. Press, Oxford, 1950] lists also a rare sign for $\frac{3}{4}$. The book continues with sections on arithmetic and algebraic problems, including several from daily life; geometry; and the possibility of Egyptian mathematics constituting a science. There are many illustrations accompanying the text.

Both parts of the book are followed by exercises, some with solutions, and a short bibliography.

The reviewer missed some interesting items of recent date which could have well been included. As evidence of the conservative nature of Egyptian mathematics, the paper, A Coptic calculation manual, by J. Drescher [Bull. Soc. d'Arch. Copte 13 (1951), 137–160], could have been cited. This dates from about A.D. 900 and is quite in the spirit of the ancient Egyptian texts. Although the golden section is not found in any of the extant texts, E. C. Kielland [Geometry in Egyptian art, Tiranti, London, 1955] advances the thesis that the golden section was used by the Egyptian artists to proportion their paintings and sculpture. A consideration of this would have been interesting.

E. B. Allen (Troy, N.Y.)

5711:

Al-Dhahir, M. W. Concerning the parallel postulate. Bull. Coll. Arts Sci., Baghdad 3 (1958), 60–65. (4 plates)

Reproduction with commentary of a text [MS order 283, Plimpton Library, Columbia University, New York], written by Athir Al-Din Al-Abhari (died 1262 or 1264 A.D.). Al-Abhari gives a "proof" of the Euclidean postulate based on the lemma that if BD bisects angle ABC, then any straight line drawn at right angles to BD will meet the containing lines AB and BC when produced. This is the reasoning which also appeared in a paper by T. Cullovin [Quart. J. Pure Appl. Math. **27** (1894), 188–190], and to which Cayley added a note on p. 225. The author, referring to Lobachevski's "Geometrical researches on the theory of parallels" [Transl. by G. B. Halsted, Univ. of Texas, Austin, 1891], theorem 23, criticizes both the reasoning of Al-Abhari and of Cayley.

D. J. Struik (Cambridge, Mass.)

5712:

Skof, Fulvia. Duplicazione del cubo secondo Archita e studio delle curve connesse al problema. I, II. Period. Mat. (4) 36 (1958), 19–40, 76–92.

"Il problema della duplicazione del cubo è riducibile a quello dell'inserzione di due medie proporzionali fra due segmenti dati. L'originalità del contributo di Archita risiede nel fatto che l'inserzione delle due medie è da lui ottenuta mediante l'intersezione di tre superficie.

Il presente lavoro consta di due parti. Nella prima esporremo una versione moderna del metodo di Archita, nella quale si seguono i metodi della geometria analitica e della geometria descrittiva; verranno inoltre studiate, dal punto di vista reale, le curve gobbe relative alla soluzione e le curve piane, loro proiezioni sui piani coordinati. Nella seconda parte si studieranno con vari metodi, dal punto di vista della geometria algebrica, le singolarità delle curve piane che si sono incontrate: esse risultano tutte razionali, ed offrono quindi un esempio di curve algebriche, dotate di singolarità di vari tipi, che scaturis-

cono dalla rizoluzione di un problema geometrix effettivo e d'importanza storica." *Dall' introduziona*

5713:

Amir-Moez, A. R. Comparison of the methods of Ibn Ezra and Karkhi. Scripta Math. 23 (1957), 173–178.

5714:

Levey, Martin. Some notes on the algebra of Abū Kāmil Shujā'; a fusion of Babylonian and Greek algebra. Enseignement Math. (2) 4 (1958), 77–92.

The greater part of this paper is given over to a presentation and comparison of the solutions by al-Khwārizmī and Abū Kāmil of the equation $x^2+21=10x$.

E. S. Kennedy (Beirut)

5715:

Thorndike, Lynn. The study of mathematics and astronomy in the thirteenth and fourteenth centuries as illustrated by three manuscripts. Scripta Math. 23 (1957), 67–76 (1958).

5716:

Kramer, Edna E. Six more female mathematicians. Scripta Math. 23 (1957), 83–95 (1958).

5717:

di Pasquale, Luigi. I cartelli di matematica disfida di Ludovico Ferrari e i controcartelli di Nicolò Tartaglia. I, II. Period. Mat. (4) 35 (1957), 253–278; 36 (1958), 175–198.

"I Cartelli inviati dal Ferrari sono sei; altrettante sono le risposte del Tartaglia. I primi due li riporteremo quasi interamente per potere determinare i limiti della contesa; degli altri citeremo solo qualche passo trascurando la parte polemica che è preponderante. Tratteremo poi più ampiamente le questioni proposte e le relative risposte. Per queste ultime oltre che dai Cartelli abbiamo attinto notizie anche dal General Trattato, ove in seguito furono inserite dal Tartaglia." *Dall' introduzione*

5718:

Gloden, A. Aperçu sur le développement des méthodes de résolution des équations algébriques. Janus 47 (1958), 73–78.

The solution of algebraic equations has a long history, having commenced in antiquity. This brief look at the solution of such equations starts with the sixteenth century during which equations of the third and fourth degrees were solved. Also, it discusses briefly the solution of the equation of the fifth degree in terms of elliptic functions and the solving of equations of the Nth degree by means of hypergeometric functions. More details on all these matters may be found in histories of mathematics and in various books on algebra.

E. B. Allen (Troy, N.Y.)

5719:

Truesdell, C. Eulers Leistungen in der Mechanik. Enseignement Math. (2) 3 (1957), 251–262.

A popular non-mathematical lecture on Euler's contributions to mechanics, delivered in Basel on the 250th anniversary of the birth of Euler.

O. Ore (New Haven, Conn.)

5720:

Biermann, Kurt-R. Iteratorik bei Leonhard Euler. Enseignement Math. (2) 4 (1958), 19–24.

The author points out that the theory of runs or sequences was already discussed by Euler in connection with the Genoa lottery. *O. Ore* (New Haven, Conn.)

5721:

Behnke, Heinrich. Otto Blumenthal zum Gedächtnis. Math. Ann. 136 (1958), 387–392.

A brief biography and a list of 38 publications.

5722:

Hodge, W. V. D. Obituary: Peter Fraser. J. London Math. Soc. 34 (1959), 111–112.

GENERAL

5723:

★Meyler, Dorothy S.; and Sutton, O. G. A compendium of mathematics and physics. D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-New York-London, 1958. 384 pp. $5.00.

The book is divided into two parts: I (pp. 1–266) Pure mathematics; II (pp. 267–372) Physics; with separate indexes. Part I has 22 sections, including arithmetic (pp. 3–9), statistics (pp. 245–247), groups, analytic and projective geometry, calculus, complex variables, and differential geometry with 7 tables (pp. 248–266); Part II has 15 sections, among which are mechanics, elasticity, viscosity, electricity and magnetism, heat, atomic physics. There are no proofs but, as the preface states, "the book is more than a mere collection of formulae, in that explanations are given as far as space permits". The authors have kept in mind both "research workers who require a reference book" and "undergraduates... preparing for examinations. The account given stops short at a point a little beyond General Honours degree standard."

BIBLIOGRAPHICAL NOTES

Analele Universitătii "C. I. Parhon" Bucureşti. Seria Acta Logica. Vol. 1, no. 1 is dated 1958 and appeared in December, 1958. The journal will contain articles on logic and on mathematical logic. It may be ordered, at the price of 10 lei per issue, from the C. I. Parhon University, Bucarest.

Estadística Española. Revista del Instituto Nacional de Estadística, Madrid. No. 1 is dated Oct.-Dec. 1958. The journal will appear every 3 months. Correspondence and subscriptions are to be sent to: I. N. de E. Ferraz, 41, Madrid. Annual subscription: 100 ptas.; single issue: 25 ptas.

Indian Journal of Mathematics. Vol. 1, no. 1 is dated Dec., 1958. The journal will be published in 2 numbers per year by the Allahabad Mathematical Society, Allahabad, and will be devoted to research papers in both mathematics and theoretical physics.

Izvestiya Akademii Nauk SSSR. Otdelenie Tehničeskih Nauk. Mehanika i Mašinostroenie. No. 1 of the new series "Mechanics and Machine-Construction" is dated Jan.-Feb. 1959.

Journal of Research of the National Bureau of Standards. Beginning July 1959 the journal will appear in four separate sections: A. Physics and Chemistry. B. Mathematics and Mathematical Physics. C. Engineering and Instrumentation. D. Radio Propagation. Sections A

and D are issued six times a year; B and D quarterly. The annual subscription for A and D is: domestic \$4.00; foreign \$4.75; for B and C: domestic \$2.25, foreign \$2.75. Subscriptions may be entered with the Superintendent of Documents, U.S. Government Printing Office, Washington 25, D.C.

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SIAM Review. Published by the Society for Industrial and Applied Mathematics, Philadelphia, Pa. Vol. 1, no. 1 is dated Jan. 1959. The journal will appear semiannually, in Jan. and July, at \$5.00 per year (individual issues \$2.50). It contains mathematics papers on an intermediate level, ideas on the training of mathematicians, problems arising from mathematics in industry and reviews of books.

The Astronomical Journal. Vol. 63, no. 10. Yale University Observatory, New Haven 11, Conn. This issue contains the proceedings of a conference on celestial mechanics, sponsored by the National Science Foundation, Columbia University and the Watson Scientific Computing Laboratory, New York, March 17-21, 1958. There are abstracts of 19 papers presented at the conference, and of a round-table discussion.

Proceedings of the VIIIth International Astronautical Congress, Barcelona, 1957. Springer-Verlag, Vienna, 1958. vii+607 pp. \$29.75. This volume contains 46 lectures, of varying length, delivered at the congress. They deal with theoretical and experimental methods; a few of them, of interest to mathematicians, are reviewed separately.

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